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TWISTORS, KÄHLER MANIFOLDS, AND BIMEROMORPHIC GEOMETRY. II

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A well-known theorem of Kodaira and Spencer [11, 15] states that any small deformation of the complex structure of a compact Kähler manifold again yields a complex manifold of Kähler type. Therefore, the question has been raised [7, 19] as to whether a similar stability result holds for compact complex manifolds that are *bimeromorphically equivalent*[1] to Kähler manifolds; that is, for manifolds of Fujiki's class $\mathscr{C}$ [6]. In this article, we will analyze the twistor spaces obtained in the previous article [13] as small deformations of the Moishezon twistor spaces discovered in [12] and show that they are generically *not* spaces of class $\mathscr{C}$, even though they are obtained as small deformations of spaces that *are*. In short, the bimeromorphic analogue of the Kodaira-Spencer stability theorem is false.[2]

In an attempt to make this article as self-contained as possible, we begin with a brief introduction to the subject, including a quick review of the essential results of the preceding article [13].

Our focus here will be on the following class of complex manifolds:

Definition 1. *A* twistor space *will herein mean a compact complex* 3*-manifold* Z *with the following properties*:

- *There is a free antiholomorphic involution* $\sigma : Z \to Z$, $\sigma^2 =$ *identity, called the* real structure *of* Z.
- *There is a foliation of* Z *by* σ*-invariant holomorphic curves* $\cong \mathbb{CP}_1$, *called the* real twistor lines.

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[1]Two connected compact complex m-manifolds X and Y are called *bimeromorphically equivalent* if there exists a complex m-manifold V, and degree 1 holomorphic maps $V \to X$ and $V \to Y$.

[2]A technically different proof of this result was found simultaneously by Campana [3], who has chosen to publish his work separately.

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- *Each real twistor line has normal bundle holomorphically isomorphic to* $\mathscr{O}(1)\oplus\mathscr{O}(1)$, *where* $\mathscr{O}(1)$ *is the degree-one line bundle on* $\mathbb{CP}_1$.

The space M of real twistor lines is thus a compact real-analytic 4-manifold, and we have real-analytic submersion $\wp : Z \to M$ known as the *twistor projection.* By a construction discovered by Roger Penrose [16], the complex structure of Z induces a half-conformally-flat conformal Riemannian metric on M, and every such metric conversely arises in this way [1]; however, we will never explicitly need this in the sequel.

We will only concern ourselves here with the class of twistor spaces admitting hypersurfaces of the following type:

Definition 2. *An* elementary divisor D *on a twistor space* Z *is a complex hypersurface* $D \subset Z$ *whose homological intersection number with a twistor line is* $+1$ *and such that* $D \cap \sigma(D) \neq \emptyset$.

An elementary divisor is necessarily a smooth hypersurface. The existence of such a divisor D is a powerful hypothesis indeed, for it follows [13, Proposition 6] that D is an n-fold blowup of $\mathbb{CP}_2$, that M is diffeomorphic to an n-fold connected sum $\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2$, and that the map $\wp|_D : D \to M$ contracts a projective line to a point, but is elsewhere an orientation-reversing diffeomorphism.

In fact, these conclusions are quite sharp.

Proposition 1. *Let* X *be any compact complex surface obtained from* $\mathbb{CP}_2$ *by blowing up distinct points. Then there exists a twistor space* Z *that contains an elementary divisor* D *such that* $D \cong X$ *as a complex surface. Moreover, given a smooth* (*respectively, real-analytic*) 1*-parameter family* X_t *of surfaces obtained from* $\mathbb{CP}_2$ *by blowing up distinct ordered points, there is a smooth* (*respectively, real-analytic*) *family* (Z_t, D_t) *of twistor spaces with elementary divisors such that* $D_t \cong X_t$.

Proof. In [12] it was shown that, given an arbitrary blowup D of $\mathbb{CP}_2$ at n collinear points, there is a twistor space Z containing a degree 1 divisor isomorphic to D. In fact, such twistor spaces Z may be explicitly constructed from conic bundles over $\mathbb{CP}_1 \times \mathbb{CP}_1$ by a process of blowing subvarieties up and down, and thus may be taken to be *Moishezon* in this case. In the accompanying article [13], the deformation theory of these twistor spaces was studied, with the following conclusion. Let $p_1 := (0, 0)$ and $p_2 := (1, 0)$ in $\mathbb{C}^2$, and let $\mathscr{W} \subset [\mathbb{C}^2]^{n-2}$ denote the set

$$\{(p_3, \dots, p_n) | p_j \in \mathbb{C}^2, p_j \neq p_k, j, k = 1, \dots, n\};$$

let $\mathscr{L} \subset \mathscr{W}$ denote the subset $p_3, \dots, p_n \in (\mathbb{C} \times \{0\})$ of collinear configurations. It was shown [13, Theorem 3] that there exists a (versal) family $(\mathscr{Z}, \mathscr{D})$ of twistor spaces with elementary divisors over a $\mathscr{U}$ neighborhood of $[\mathscr{L} \times (\mathbb{R}^+)^n] \subset [\mathscr{W} \times (\mathbb{R}^+)^n]$ such that the divisor D associated with a configuration of points $p_1, \dots, p_n \in \mathbb{C}^2 \subset \mathbb{CP}_2$ and an arbitrary collection of positive weights $m_1, \dots, m_n \in \mathbb{R}^+$ is isomorphic to $\mathbb{CP}_2$ blown up at $p_1, \dots, p_n$.

Now suppose we are given an arbitrary compact complex surface X obtained from $\mathbb{CP}_2$ by blowing up n distinct points $q_1, \dots, q_n$. There is a line $L \subset \mathbb{CP}_2$ that misses $q_1, \dots, q_n$, and we now identify $\mathbb{CP}_2 - L$ with $\mathbb{C}^2$ in such a way that $q_1 = (0, 0)$ and $q_2 = (1, 0)$. Assign all the points, say, weight 1. By making a linear transformation, we may also take the points $q_1, \dots, q_n$ to be as close as we like to the z_1-axis, so that our configuration becomes a point of $\mathscr{U}$. The corresponding fiber of our family $(\mathscr{Z}, \mathscr{D})$ then comes equipped with an elementary divisor isomorphic to the given X.

On the other hand, suppose we are instead given an arbitrary smooth family X_t of surfaces obtained by blowing up n distinct, ordered points in $\mathbb{CP}_2$, where t ranges over $\mathbb{R}$. Let $\mathscr{X} \to \mathbb{R}$ denote the family with fibers $\{X_t\}$. There is a bundle $\mathscr{P} \to B$ of $\mathbb{CP}_2$'s from which $\mathscr{X} \to \mathbb{R}$ is obtained by blowing up n sections $q_1, \dots, q_n$; let $\mathscr{P}^* \to \mathbb{R}$ denote the bundle of dual planes, in which the $q_1, \dots, q_n$ define n complex hypersurfaces. The complement of these hypersurfaces in $\mathscr{P}^*$ has real codimension 2, so, by transversality, a generic smooth (respectively, real-analytic) section of $\mathscr{P}^*$ will miss them, and we may therefore smoothly (respectively, real-analytically) choose a projective line L_t in each fiber P_t of $\mathscr{P}$ that misses the points $q_{1_t}, \dots q_{n_t}$. Using q_1 as the zero section, the complement of these chosen lines becomes a vector bundle over $\mathbb{R}$ and so may be trivialized in such a manner that $q_2 \equiv (1, 0)$. Our family of surfaces, therefore, may be thought of as associated with a family of point configurations $(q_1, \dots, q_n)_t$ in $\mathbb{C}^2$, where $q_1 \equiv (0, 0)$ and $q_2 \equiv (1, 0)$. Again, let us assign each point a positive weight, say 1. Now there is a positive real-analytic function $F(\zeta_3, \dots, \zeta_n)$ such that a weighted configuration $((0, 0, 1), (0, 1, 1), (\zeta_3, \eta_3, 1), \dots, (\zeta_n, \eta_n, 1))$ is in $\mathscr{U}$ provided that

$$\sum |\eta_j|^2 < F(\zeta_3, \dots, \zeta_n).$$

Setting $(q_3, \dots, q_n)_t = ((\zeta_3(t), \eta_3(t)), \dots, (\zeta_n(t), \eta_n(t)))$, define

$$(p_1, \dots, p_n)_t := \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\dfrac{F(\zeta_3(t), \dots, \zeta_n(t))}{(1 + \sum |\zeta_j(t)|^2)}} \end{bmatrix} (q_1, \dots, q_n)_t .$$

The family $((p_1, 1), \dots, (p_n, 1))_t$ of weighted configurations then takes values within the parameter space $\mathscr{U}$ of the family $(\mathscr{Z}, \mathscr{D})$. Pulling back $(\mathscr{Z}, \mathscr{D})$ now yields the desired family of twistor spaces with elementary divisors. □

It might be emphasized, incidentally, that the twistor space Z is by no means determined by the intrinsic structure of a elementary divisor D. Nonetheless, we will presently see that the intrinsic structure of such a divisor *does* tell us a great deal about a twistor space, and, in particular, is sufficient to determine its *algebraic dimension*.

Let us recall that the algebraic dimension $a(Z)$ of a compact complex manifold Z is by definition the degree of transcendence of its field of meromorphic functions, considered as an extension of the field $\mathbb{C}$ of constant functions. Equivalently, the algebraic dimension of Z is precisely the maximal possible dimension of the image of Z under a meromorphic map to $\mathbb{CP}_N$; in particular, $a(Z) \leq \dim_{\mathbb{C}}(Z)$. When equality is achieved in the latter inequality, Z is said to be a *Moishezon manifold* [14] and a suitable sequence of blowups of Z along complex submanifolds will then result in a projective variety.

The following lemma of Campana will be of critical importance.

Lemma 1 [2]. *A twistor space Z is bimeromorphic to a Kähler manifold iff it is Moishezon.*

Proof. Let p and q be distinct points of a real twistor line L in a twistor space Z, and let S_p (respectively, S_q) denote the space of rational curves through p (respectively, q) that are deformations of L. Assume that Z is in the class $\mathscr{C}$. Because the components of the Chow variety of Z are therefore compact, the correspondence space

$$Z' := \{(r, C_1, C_2) \in Z \times S_p \times S_q \mid r \in C_1 \cap C_2\}$$

is thus a compact complex space; by blowing up any singularities, we may assume that Z' is smooth. But since a real twistor line has the same normal bundle as a projective line in $\mathbb{CP}_3$, a generic point of Z is joined to either p or q only by a discrete set of curves of the fixed class. The correspondence space Z', therefore, is generically a branched cover of Z, and, in particular, is a 3-fold. On the other hand, we have a canonical map

$$\phi : Z' \to \mathbb{P}(T_pZ) \times \mathbb{P}(T_pZ) \cong \mathbb{CP}_2 \times \mathbb{CP}_2$$

obtained by taking the tangent spaces of curves at their base points p or q. Let r be a point of Z that is not on L, but close enough to L so that r is joined to p and q by small deformations C_1 and C_2 of L, both of which are $\mathbb{CP}_1$'s with normal bundle $\mathscr{O}(1) \oplus \mathscr{O}(1)$. Then (r, C_1, C_2) is a point of Z' at which the derivative of ϕ has maximal rank. Pulling back meromorphic functions from $\mathbb{CP}_2 \times \mathbb{CP}_2$ to Z', therefore, must yield three algebraically independent functions, and Z', therefore, is a Moishezon space. But since the projection $Z' \to Z$ is surjective, and since the class of Moishezon manifolds is closed under holomorphic surjections [14], it follows that Z is also a Moishezon manifold.

The converse, of course, is trivial. □

On the other hand, the following lemma allows one to determine the algebraic dimension of a twistor space.

Lemma 2 [17]. *Any meromorphic function on a simply connected twistor space Z can be expressed as the ratio of two holomorphic sections of a sufficiently large power κ^{-m} of the anticanonical line bundle $\kappa^{-1} := \bigwedge^3 TZ$.*

Proof. We begin by observing that any (compact) twistor space satisfies $h^1(Z, \mathscr{O}) = b_1(Z)$. This is a consequence of the *Ward correspondence* [20], which says that the set of holomorphic vector bundles on Z that are trivial on real twistor lines is in 1-1 correspondence with the instantons on M; in particular, every holomorphic line bundle on Z with $c_1 = 0$ is obtained by pulling back a flat $\mathbb{C}_*$-bundle from M and equipping it with the obvious holomorphic structure. With the exponential sequence

$$\cdots \to H^1(Z, \mathscr{O}) \to H^1(Z, \mathscr{O}_*) \xrightarrow{c_1} H^2(Z, \mathbb{Z}) \to \cdots,$$

this implies that holomorphic line bundles on a simply connected twistor space are classified by their Chern classes.

Since we have assumed that Z is simply connected, it follows that $H^2(Z, \mathbb{Z})$ is free. On the other hand, the Leray-Hirsch theorem tells us that $H^2(Z, \mathbb{Q}) = \mathbb{Q}c_1(Z) \oplus H^2(M, \mathbb{Q})$. The latter splitting of the cohomology is exactly the decomposition of $H^2(Z, \mathbb{Q})$ into the (∓ 1)-eigenspaces of σ^*; a class will be called *real* if it is in the (-1)-eigenspace, and a complex line bundle will be called real if its first Chern class is real. There is thus a unique "fundamental" holomorphic line bundle ξ on Z such that any real holomorphic line bundle is a power of ξ and such that the restriction of ξ to a twistor line is positive; in particular, $\kappa = \xi^k$ for some k. While we will not need to know this explicitly, it can in fact be shown [9] that $k = 4$ if M is spin, $k = 2$ otherwise.

Now suppose that we are given a meromorphic function f on such a Z. The function f can a priori be expressed in the form $f = g/h$, where g and h are holomorphic sections of a line bundle $\eta \to Z$; for example, we could take η to be the divisor line bundle of the polar locus of f. The pullback $\sigma^*\overline{\eta}$ of the conjugate line bundle of η is automatically holomorphic and $\sigma^*\overline{g}$ and $\sigma^*\overline{h}$ are holomorphic sections of this bundle. The holomorphic bundle $\eta \otimes \sigma^*\overline{\eta}$ is now *real* and has sections, and so must be of the form ξ^m for some positive integer m. Thus

$$f = \frac{gh^{k-1}\sigma^*\overline{h}^k}{h^k\sigma^*\overline{h}^k}$$

expresses our meromorphic function as the quotient of two holomorphic sections of κ^m. □

We have already seen that there are examples of Moishezon twistor spaces Z containing an elementary divisor D isomorphic to $\mathbb{CP}_2$ blown up at a collinear configuration of points. We will now see that the situation is dramatically different when the intrinsic structure of D is generic.

Proposition 2. *Suppose that Z is a twistor space with an elementary divisor isomorphic to the blowup of $\mathbb{CP}_2$ at n generic points, $n \geq 7$. Then Z has no nonconstant meromorphic functions, and so has algebraic dimension 0. The set of configurations $(p_1, \ldots, p_n)$ of points in $\mathbb{CP}_2$ that are generic in this sense is the complement of a countable union of proper algebraic subvarieties of $(\mathbb{CP}_2)^n$ and, in particular, has full measure.*

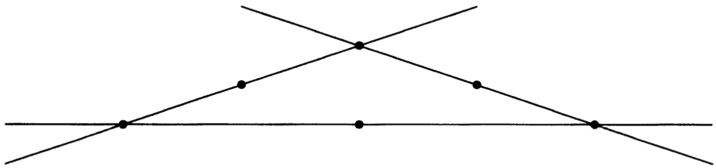

FIGURE 1

Proof. Let us begin by considering the case of a configuration of n points in $\mathbb{C}^2$ containing a 6-point configuration of the type shown in Figure 1.

We assume that the other points of the configuration are not on any of three projective lines of the figure. The proper transforms of these three lines are then (-2)-curves E_j, $j = 1, 2, 3$. The anticanonical bundle κ_D^{-1} of the surface D thus satisfies $\kappa_D^{-1}|_{E_j} \cong \mathcal{O}$. On the other hand, since the half-anticanonical bundle of Z is given by $\kappa^{-1/2} = [D] \otimes [\overline{D}]$, we have

$$\kappa^{-1/2}|_D = \nu \otimes [L_\infty] ,$$

where ν denotes the normal bundle of $D \subset Z$ and $L_\infty \subset D$ is the projective line $D \cap \overline{D}$. Yet the adjunction formula yields

$$\kappa^{-1}|_D = \nu \otimes \kappa_D^{-1} ,$$

so that $\nu^2 \otimes [L_\infty]^2 = \nu \otimes \kappa_D^{-1}$, implying that $\nu = \kappa_D^{-1} \otimes [L_\infty]^{-2}$ and hence

$$\kappa^{-1/2}|_D = \kappa_D^{-1} \otimes [L_\infty]^{-1} . \tag{1}$$

It follows that

$$\kappa^{-1/2}|_{E_j} \cong \mathcal{O}(-1) .$$

On the other hand, the normal bundle N_j of $E_j \subset D$ is isomorphic to $\mathcal{O}(-2) \to \mathbb{C}P_1$. Since

$$\begin{aligned}\Gamma(E_j, \mathcal{O}((\kappa^{-m/2}|_{E_j}) \otimes N_j^{-k})) &= \Gamma(\mathbb{C}P_1, \mathcal{O}(-m+2k)) \\ &= 0 \quad \text{if } k < \tfrac{m}{2} ,\end{aligned}$$

it follows that any section of $\kappa^{-m/2}|_D$ vanishes along E_j to order $[\frac{m-1}{2}]$. But through the generic point of D we can find a projective line in D passing through a blown-up point not on the diagram, avoiding all other blown-up points, and meeting the E_j in three distinct points. Letting L denote the proper transform of such a line, one has

$$\begin{aligned}\kappa^{-1/2}|_L &= (\kappa_D^{-1} \otimes [L_\infty]^{-1})|_L \\ &\cong \mathcal{O}(2) \otimes \mathcal{O}(-1) \\ &\cong \mathcal{O}(1) ,\end{aligned}$$

so that $\kappa^{-m/2}|_L \cong \mathcal{O}(m)$. Yet any holomorphic section of of $\kappa^{-m/2}|_D$ must have 3 zeroes on L of multiplicity $[\frac{m-1}{2}]$ at $L \cap E_j$. Since $3[\frac{m-1}{2}] > m$ for

$m > 6$, we conclude that such a section must vanish identically on L provided m is sufficiently large. Hence $\Gamma(D, \mathcal{O}(\kappa^{-m/2})) = 0$ for m sufficiently large, and hence, by taking tensor powers of sections, for all $m > 0$. Similarly, $\Gamma(\overline{D}, \mathcal{O}(\kappa^{-m/2})) = 0$ for all $m > 0$. From the exact sequences

$$(2) \qquad 0 \to \mathcal{O}_Z(\kappa^{-(m-1)/2}) \to \mathcal{O}_Z(\kappa^{-m/2}) \to \mathcal{O}_{D\cup\overline{D}}(\kappa^{-(m-1)/2}) \to 0,$$

we conclude by induction that $\Gamma(Z, \mathcal{O}(\kappa^{-m/2})) = \mathbb{C}$ for all $m > 0$. By Lemma 2, any meromorphic function on Z must, therefore, be constant.

We now examine the case of D obtained from $\mathbb{CP}_2$ by blowing up $n > 6$ generically located points. For each n-tuple of points $(p_1, \dots, p_n)_u$ in $\mathbb{C}^2 = \mathbb{CP}_2 - L_\infty$, let D_u denote the corresponding blowup of $\mathbb{CP}_2$, and consider the behavior of $h^0(D_u, \mathcal{O}(\kappa_{D_u}^{-m} \otimes [L_\infty]^{-m}))$. By the semicontinuity principle [8] and the above calculation, this vanishes, for m fixed, on a nonempty Zariski-open subset of configurations. The set of n-point configurations for which $h^0(D_u, \mathcal{O}(\kappa_{D_u}^{-m} \otimes [L_\infty]^{-m})) \neq 0$ for some m, therefore, is a countable union of subvarieties, and so has measure 0. Using the exact sequence (2) and the isomorphism (1), we conclude that $\Gamma(Z, \mathcal{O}(\kappa^{-m/2})) = \mathbb{C}\ \forall m \geq 0$ provided that Z contains an elementary divisor D obtained from $\mathbb{CP}_2$ by blowing up $n > 6$ generic points. Again applying Lemma 2, we conclude that, for $n \geq 7$, any meromorphic function on a twistor space Z containing a generic elementary divisor, therefore, must be constant. □

Our main result now follows immediately.

Theorem 1. *The class $\mathscr{C}$, consisting of compact complex manifolds that are bimeromorphic to Kähler manifolds, is not stable under small deformations.*

Proof. By Propositions 1 and 2, there exist 1-parameter families of twistor spaces Z_t for which almost every Z_t has algebraic dimension 0, whereas Z_0 is Moishezon; in fact, it suffices to take Z_0 to be one of the explicit examples of [12], with D_0 corresponding to a collinear configuration of $n \geq 7$ points, arrange for the curve of configurations $(p_1, \dots, p_n)_t$ to be real-analytic and contain at least one generic configuration. (Actually, one can do better: by taking the elementary divisors D_t to all correspond to configurations containing projective copies of Figure 1 when $t \neq 0$, one can even arrange for Z_0 to be the *only* Moishezon space in the family.) By Lemma 1, the non-Moishezon twistor spaces of the family Z_t are not of class $\mathscr{C}$, despite the fact that they are arbitrarily small deformations of the Moishezon space Z_0. □

Remarks. (i) In order to keep this article as short and clear as possible, we only considered the case of $n \geq 7$ and only presented the extreme cases of $a(Z) = 3$ and $a(Z) = 0$. In fact, in can be shown that generically $a(Z) < 3$ as soon as $n \geq 4$. One can also find simple noncollinear configurations for which $a(Z) = 1, 2$ as soon as $n \geq 5$. Finally, one can show that the existence of an elementary divisor corresponding to a collinear configuration *forces* Z to be one of the examples of [12], and, in particular, Moishezon. For details, see [18].

(ii) The existence of self-dual metrics on arbitrary connected sums $\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2$ was first proved abstractly by Donaldson and Friedman [4] and, using completely different methods, by Floer [5]. Unlike the methods used here, these methods do not show that the twistor space of some of these metrics are Moishezon. It was nonetheless the Donaldson-Friedman construction that originally gave the authors reason to believe that the generic deformation of the explicit twistor spaces of [12] should not be of Fujiki-class $\mathscr{C}$. For providing this source of inspiration, as well as for their friendly advice and encouragement, the authors, therefore, would like to thank Robert Friedman and Simon Donaldson.

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ABSTRACT. Using examples [13] of compact complex 3-manifolds that arise as twistor spaces, we show that the class of compact complex manifolds bimeromorphic to Kähler manifolds is not stable under small deformations of complex structure.

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