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The Einstein–Weyl equations in complex and quaternionic geometry*

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Abstract: Einstein–Weyl manifolds with compatible complex structures are shown to be given as torus bundles on Kähler–Einstein manifolds, extending known results on locally conformal Kähler manifolds. The Weyl structure is derived from a Ricci-flat metric constructed by Calabi on the canonical bundle of the Kähler–Einstein manifold. Similar questions are addressed when the Weyl geometry admits compatible hypercomplex or quaternionic structures.

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1. Introduction

Einstein–Weyl spaces are conformal manifolds satisfying a conformally invariant version of the Einstein equations. In this note we consider Einstein–Weyl manifolds with additional complex or quaternionic structures. In the complex case this rapidly connects with work of Vaisman [18, 19] on locally conformal Kähler manifolds and generalised Hopf manifolds: a conformal manifold with compatible complex structure and connection is the same as a locally conformal Kähler manifold when the dimensions are at least six; and when the manifolds are compact, imposing the Einstein–Weyl

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equations gives a generalised Hopf structure which is locally conformal to Kähler–Einstein. These relationships are discussed in detail in the first two sections. We then go on to consider the further implications of the Einstein–Weyl equations for these manifolds showing that under certain regularity conditions they are given as torus bundles over Kähler–Einstein manifolds of positive scalar curvature and that their conformal structure is derived from a Ricci-flat metric constructed by Calabi on the canonical bundle of such a manifold. This extends the result of Vaisman that without the Einstein condition these manifolds are torus bundles over Kähler manifolds. These results lead to a quotient construction for Hermitian–Einstein–Weyl manifolds and a classification of the compact strongly regular six-dimensional examples. We also make some remarks about non-regular manifolds.

In the last section we consider similar questions when a Weyl space admits compatible quaternionic structures. We show that these are often locally conformal to hyperKähler or quaternionic Kähler, in which case these spaces are automatically Einstein–Weyl, and give a classification of those that are locally conformal to hyperKähler.

2. Hermitian–Weyl manifolds

Let M be a compact manifold with conformal class $[g]$ and a Weyl connection D. By definition, D is a torsion-free connection preserving the conformal structure, so choosing a representative metric g gives a 1-form ω defined by $Dg = \omega \otimes g$. Suppose M admits a compatible almost complex structure I, that is $DI = 0$ and $g(IX, IY) = g(X, Y)$ for all tangent vectors X, Y, then we shall call M a *Hermitian–Weyl* space.

Hermitian–Weyl manifolds are closely related to Kähler manifolds. Since D is torsion-free, $DI = 0$ implies that I is integrable and if $\Omega_I(X, Y) = g(X, IY)$ is the corresponding fundamental 2-form then $d\Omega_I = \omega \wedge \Omega_I$. Multiplication by Ω_I^{n-2} gives an isomorphism between $\Lambda^2 T^*M$ and $\Lambda^{2n-2}T^*M$, so if M has dimension $2n \geqslant 6$, multiplication by Ω_I as a map from $\Lambda^2 T^*M$ to $\Lambda^4 T^*M$ is injective. This implies that $d\omega = 0$, since

$$0 = d^2\Omega_I = d(\omega \wedge \Omega_I) = d\omega \wedge \Omega_I.$$

Thus locally we may write $\omega = df$. Now multiplying g by e^{-f} gives a representative $\hat{g}$ for which $\hat{\omega} = \omega - df = 0$ and $d\hat{\Omega}_I = 0$. Thus M is locally conformal Kähler. (If M is simply connected then this may be done globally and M is conformal to Kähler.)

Conversely let (N, g, I) be a locally conformal Kähler manifold of real dimension at least 4. Then if $U \subset N$ is a sufficiently small open set there exists a function f_U such that $e^{-f_U}g$ is Kähler. In particular, $e^{-f_U}\Omega_I$ is closed, so $d\Omega_I = df_U \wedge \Omega_I$. On $U \cap V$, we have $(df_U - df_V) \wedge \Omega_I = 0$ and, since Ω_I is non-degenerate, this implies that the 1-form $\omega = df_U$ is well-defined globally. If ∇ is the Levi–Civita connection of g and $\omega^\sharp$ is the vector field dual to ω then one obtains a Weyl connection D with $Dg = \omega \otimes g$ by defining $D = \nabla + a$, where

$$2a_X Y = g(X, Y)\omega^\sharp - \omega(X)Y - \omega(Y)X$$

(see [13]). However, on U, the connection D is equal to $\hat{\nabla}$, the Levi–Civita connection of the Kähler metric $\hat{g} = e^{-f_U} g$, so D also preserves I.

Proposition 2.1 (Vaisman). *Any Hermitian–Weyl manifold of (real) dimension at least* 6 *is locally conformal Kähler. Conversely, a locally conformal Kähler manifold of dimension at least* 4 *admits a compatible Hermitian–Weyl structure.* □

For future reference, note the following:

Lemma 2.2. *Let* $(M, [g], I, D)$ *be a Hermitian–Weyl manifold with representative metric* g. *Let* ω *be the corresponding* 1*-form and let* ∇ *be the Levi–Civita connection of* g. *If the vector field* X *is either* $\omega^\sharp$ *or* $I\omega^\sharp$, *then* $\nabla_X I = 0$.

Proof. The result follows by noting that $D = \nabla + a$ as above and checking that $(a_X I)Y = a_X(IY) - I a_X(Y)$ is zero in these two cases. □

3. Hermitian–Einstein–Weyl manifolds

Suppose M is a Weyl manifold. The conformally invariant analogue of the Einstein condition is that the symmetrised Ricci tensor of D be a multiple of g at each point of M. Such a structure is called *Einstein–Weyl.* For a compact Weyl manifold Gauduchon [7] showed that, up to homothety, there is a unique choice of metric g in the conformal class such that the corresponding 1-form ω is co-closed. If M is Hermitian–Weyl and has dimension $2n \geqslant 6$ then we saw above that ω is always closed, so for Gauduchon's choice of metric, the corresponding 1-form ω is harmonic. Another result of Gauduchon [8] says that on an Einstein–Weyl manifold if g is a compatible metric such that ω is harmonic then ω is parallel with respect to the Levi–Civita connection of g. Thus on a Hermitian–Einstein–Weyl manifold of dimension at least 6 we may choose a representative metric such that ω is parallel and $\omega^\sharp$ is a Killing vector field. In this situation the Einstein–Weyl equations take the form

$$r^\nabla + \frac{n-2}{4}\omega \otimes \omega = \lambda\, g,$$

where λ is a function on the n-dimensional manifold M and r^∇ is the Ricci tensor of Gauduchon's metric g (see [13]). Note that $d\|\omega\|^2 = 0$, so $\|\omega\|$ is constant and, assuming g is not Einstein, we may rescale the metric by a homothety so that this constant is 1. In Vaisman's terminology, a Hermitian–Weyl manifold with ω parallel is a generalised Hopf structure. For this choice of metric we have:

Lemma 3.1 (Vaisman). *The vector fields* $\omega^\sharp$ *and* $I\omega^\sharp$ *are commuting holomorphic Killing vectors.*

Proof. We have already shown that $\omega^\sharp$ is Killing. Using the lemma in the previous section one has

$$\begin{aligned}(L_{\omega^\sharp}I)X &= [\omega^\sharp, IX] - I[\omega^\sharp, X]\\ &= \nabla_{\omega^\sharp}(IX) - \nabla_{IX}\omega^\sharp - I\nabla_{\omega^\sharp}X + I\nabla_X\omega^\sharp\\ &= \nabla_{\omega^\sharp}(IX) - I\nabla_{\omega^\sharp}X = (\nabla_{\omega^\sharp}I)X = 0.\end{aligned}$$

So $\omega^\sharp$ is holomorphic and taking $X = \omega^\sharp$ we see that $\omega^\sharp$ commutes with $I\omega^\sharp$. Also the integrability of I implies $L_{I\omega^\sharp}I = IL_{\omega^\sharp}I$, so $I\omega^\sharp$ is holomorphic. Now for any X, $(L_X\Omega_I)(Y,Z) = (L_Xg)(Y,IZ) + g(Y,(L_XI)Z)$, so

$$L_{I\omega^\sharp}\Omega_I = d(I\omega^\sharp \lrcorner \Omega_I) + I\omega^\sharp \lrcorner d\Omega_I = d\omega + I\omega^\sharp \lrcorner (\omega \wedge \Omega_I) = 0$$

implies that $I\omega^\sharp$ is also Killing. $\square$

Consider the foliation $\mathcal{E}$ generated by the vector fields $\omega^\sharp$ and $I\omega^\sharp$.

Proposition 3.2. *Let M be a compact Hermitian–Einstein–Weyl manifold which is not globally conformal to Einstein and has dimension $2n \geqslant 6$. If the leaves of $\mathcal{E}$ are two-dimensional and closed then $N = M/\mathcal{E}$ is a Riemannian orbifold which, at regular points, is Kähler–Einstein with positive scalar curvature.*

Proof. That $N = M/\mathcal{E}$ is an orbifold follows from general results on Riemannian foliations (for example see [12]).

By the lemma in the previous section, $\nabla_{\omega^\sharp}\omega^\sharp$, $\nabla_{I\omega^\sharp}\omega^\sharp$, $\nabla_{\omega^\sharp}I\omega^\sharp$ and $\nabla_{I\omega^\sharp}I\omega^\sharp$ all vanish and so the leaves of $\mathcal{E}$ are totally geodesic with respect to the Levi–Civita connection.

At each point x of M define a horizontal subspace $\mathcal{H}_x \subset T_xM$ as the orthogonal complement of the complex linear span of $\omega^\sharp$. Let θ be the 1-form $-\omega \circ I$ and define a symmetric 2-tensor by $h = g - \omega^2 - \theta^2$. Now $h(\omega^\sharp, X) = g(\omega^\sharp, X) - \omega(X) = 0$ and $h(I\omega^\sharp, X) = g(I\omega^\sharp, X) + \theta(X) = 0$, so h is a non-degenerate metric on $\mathcal{H}$. Also, h is preserved by the vector fields $\omega^\sharp$ and $I\omega^\sharp$, since these vector fields are Killing,

$$\begin{aligned}L_{\omega^\sharp}\omega &= \omega^\sharp \lrcorner d\omega = 0\\ L_{\omega^\sharp}\theta &= 2\nabla_{\omega^\sharp}\theta = 2(\nabla_{\omega^\sharp}I\omega^\sharp)^\flat = 0\end{aligned}$$

and similarly for $I\omega^\sharp$. Thus h descends to a Hermitian metric on $N = M/\mathcal{E}$. Furthermore the fundamental 2-form ψ of N pulls-back to $\Psi = \Omega_I + \omega \wedge \theta$ and $d\Psi = \omega \wedge \Omega_I - \omega \wedge d\theta$ which is zero on $\mathcal{H}$, so ψ is closed and N is Kähler.

We claim that Ψ is just $d\theta$. Firstly, $d\theta$ is horizontal, since if X is either $\omega^\sharp$ or $I\omega^\sharp$ then $X \lrcorner d\theta = L_X\theta - d(X \lrcorner \theta) = 0$. Suppose $X \in \mathcal{H}$, then

$$d\theta(X, IX) = \omega(I[X, IX]) = -\omega(D_XX) - \omega(D_{IX}IX).$$

Now for any $Y \in \mathcal{H}$,

$$\omega(D_YY) = \omega(\nabla_YY) + \omega(a_YY) = -(\nabla_Yg)(\omega^\sharp, Y) + \tfrac{1}{2}g(Y,Y),$$

so $d\theta(X, IX) = -\frac{1}{2}h(X,X) - \frac{1}{2}h(IX, IX) = \Psi(X, IX)$.

To show h is Einstein at regular points we work on an small open set U of M for which the quotient P of U by the action generated by $\omega^\sharp$ is a manifold. Then U is a flat bundle over P and the bundle P over N has curvature $d\theta = \Psi$. Now U has one-dimensional totally geodesic fibres and the Weyl form ω has unit length so the Einstein–Weyl equations [13] give

$$r_P = \tfrac{1}{2}(n-1)g_P,$$

where $g_P = h + \theta^2$ is the induced metric on P and r_P is its Ricci curvature. Thus P is Einstein with positive scalar curvature. With respect to the fibration $P \to N$, the Einstein equations are equivalent to

$$\begin{aligned}
&\|\psi\|^2 = 4\Lambda,\\
&d^*\psi = 0,\\
&r_N(X,Y) - \tfrac{1}{2}(\psi_X, \psi_Y) = \Lambda h(X,Y),
\end{aligned}$$

where r_N is the Ricci curvature of h, $\Lambda = \frac{1}{2}(n-1)$ is the Einstein constant of g_P and for an orthonormal basis $\{X_i\}$, $(\psi_X, \psi_Y) = \sum_i \psi(X, X_i)\psi(Y, X_i)$. Also $(\psi_X, \psi_Y) = \sum_i h(IX, X_i)h(IY, X_i) = h(X,Y)$, so

$$r_N = \frac{n}{2}h$$

and N is Einstein with positive scalar curvature. □

The simplest example of this construction is obtained by taking $M = S^1 \times S^{2n-1}$ regarded as a quotient of $\mathbb{C}^n \setminus \{0\}$ modulo the action of $\mathbb{Z}$ generated by $v \mapsto 2v$. This gives M a Hermitian–Einstein–Weyl structure which is not conformally Einstein (globally). The vector field $\omega^\sharp$ is a generator for the trivial circle bundle over $P = S^{2n-1}$ and the quotient $N = M/\mathcal{E}$ is complex projective space $\mathbb{C}\mathrm{P}(n-1)$ with its Fubini-Study metric. Further examples will be found in the sections below.

Remarks. (1) Note that the above proof also shows that the symmetrised Ricci tensor of the Einstein–Weyl structure is necessarily zero, so the local Kähler metrics are Ricci-flat. This is a special case of a result of Gauduchon [8]. This would not have been apparent if we had proved the above result by using a Kähler quotient construction on the universal cover of M together with results of Futaki [6], however this second approach will be useful later in the hyperHermitian case.

(2) If the leaves of $\mathcal{E}$ are diffeomorphic and closed, then we will say that M is *regular*. We will call M *strongly regular* if M is regular and the orbits of $\omega^\sharp$ are closed. Examples of regular Hermitian–Einstein–Weyl spaces which are not strongly regular are obtained by taking the quotient of $\mathbb{C}^n \setminus \{0\}$ by the action of $\mathbb{Z}$ generated by $v \mapsto 2e^{i\pi\alpha}v$, where α is irrational. Similarly, non-regular Hermitian–Einstein–Weyl spaces may be obtained by dividing $\mathbb{C}^n \setminus \{0\}$ by the action of $\mathbb{Z}$ generated by $v = (v_1, \dots, v_r) \mapsto 2(e^{i\pi\alpha_1}v_1, \dots, e^{i\pi\alpha_r}v_r)$, where $\mathbb{C}^n = \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_r}$, $r > 1$, and the α_i are distinct real numbers with at least two of them linearly independent over $\mathbb{Q}$.

(3) If M is a compact Hermitian–Weyl manifold which is not regular it is still possible to describe some of the structure of M. As before choose a representative metric for which ω is parallel. Now let K_0 be the group of isometries generated by $\omega^\sharp$ and $I\omega^\sharp$. Since M is compact, the group of isometries of M is compact and so the closure K of K_0 is also compact. The elements of the Abelian group K act isometrically, holomorphically and they preserve the closed 2-form $\Psi = \Omega_I + \omega \wedge \theta$. One may now construct a moment map $\mu\colon M \to \mathfrak{k}^*$ such that for each vector field X generated by the action of K

$$d\langle \mu, X\rangle = X \lrcorner \Psi,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between the Lie algebra $\mathfrak{k}$ and its dual. This map commutes with the action of K and generically has rank $\dim K - 2$. For a generic a in the image of μ, the preimage $\mu^{-1}(a)$ contains K-orbits on which the action of K is locally free. Following the usual proof of the Kähler quotient construction [9] one sees that $\mu^{-1}(a)/K$ is a (usually singular) Kähler manifold of dimension $\dim M - 2\dim K + 2$. For the non-regular examples above this quotient is $\mathbb{C}\mathrm{P}(n_1 - 1) \times \cdots \times \mathbb{C}\mathrm{P}(n_r - 1)$ with the product of the Fubini-Study metrics.

(4) Examples where the quotient $N = M/\mathcal{E}$ has orbifold singularities are easily produced from Lens spaces. Let M' be $S^1 \times S^{2n+1}$ with its standard Hermitian–Einstein–Weyl structure, choose integers $p, s_1, \ldots, s_n$ which are coprime and consider the $\mathbb{Z}_p$-action generated by $(z, q) \mapsto (z, \mathrm{diag}(e^{2\pi i/p}, e^{2\pi s_1 i/p}, \ldots, e^{2\pi s_n i/p})q)$, where $z \in S^1$ and $q \in S^{2n+1} \subset \mathbb{C}^{n+1}$. This action is free on M' and the quotient manifold $M = M'/\mathbb{Z}_p$ inherits a Hermitian–Einstein–Weyl structure. However, the induced $\mathbb{Z}_p$-action on $\mathbb{C}\mathrm{P}(n)$ is not free, in general, and so $N = M/\mathcal{E}$ is an orbifold.

4. Inverse construction

In this section, we relate compact strongly regular Hermitian–Einstein–Weyl structures on a manifold of dimension at least six, to the following metric:

Proposition 4.1. *Away from the zero-section, the canonical bundle of a Kähler–Einstein manifold of positive scalar curvature admits a Ricci-flat Kähler metric which transforms homothetically under the fibrewise action of $\mathbb{C}^*$ and which is homothetic to the base metric in the directions orthogonal to the fibres. Furthermore, this metric is unique up to homothety.*

Proof. Using the formalism of [15], let (N, h) be the Kähler–Einstein $2n$-manifold of positive scalar curvature. Suppose $\pi\colon F \to N$ is the principal $U(n)$-bundle of unitary frames, so F_x consists of complex linear isometries $u\colon \mathbb{C}^n \to T_xN$. Define a 1-form ζ on F with values in $\mathbb{C}^n$ by $\zeta_u(v) = u^{-1}(\pi_* v)$, when $u \in F$ and $v \in T_uF$. Let $\omega = \omega_+ + \omega_- \in \Omega^1(F, \mathfrak{su}(n) \oplus \mathfrak{u}(1))$ be the connection form. Then the first structure equation gives

$$d\zeta = -\omega_+ \wedge \zeta - \omega_- \wedge \zeta.$$

Let $\mathcal{K}$ be the $U(n)$-representation $\Lambda^n\mathbb{C}^n$, where $\mathbb{C}^n$ is the standard $U(n)$ module, then the canonical bundle is $K = F\times_{U(n)}\mathcal{K}$. On $F\times\mathcal{K}$, let z be the projection onto the second factor. Omitting pull-back signs, for $A\in U(n)$ we have $R_A^*\zeta = \bar{A}^t\zeta$ and $R_A^*dz = (\det A)^{-1}dz$. Define α to be $dz - z\omega_-$ and let $r^2 = z\bar{z}$. Then $d\alpha = -\alpha\wedge\omega_- - zd\omega_-$, but $d\omega_-$ is the curvature of the canonical bundle and N is Kähler–Einstein, so $d\omega_- = \lambda\bar{\zeta}^t\wedge\zeta$ for some constant λ which is a positive multiple of the scalar curvature. As in [15], putting

$$g = f(r^2)\,\mathrm{Re}(\alpha\otimes\bar{\alpha} + \mu r^2\bar{\zeta}^t\otimes\zeta),$$
$$\nu = f(r^2)(\alpha\wedge\bar{\alpha} + \mu r^2\bar{\zeta}^t\wedge\zeta)$$

defines a metric and a non-degenerate 2-form on $K\backslash 0$ whenever μ is a positive constant and any metric satisfying the hypotheses of the proposition must have this form. Now g and ν define an almost complex structure I. As in [13], the condition that I be integrable is that the curvature $d\omega_-$ has no $(0,2)$-component, which is automatic in this case since $d\omega_- = \lambda\bar{\zeta}^t\wedge\zeta$ is of type $(1,1)$. The condition that ν be closed reduces to

$$f = c\,r^{2(-1+\lambda/\mu)},$$

for some constant c which we may take to be 1.

To decide when the metric is Ricci-flat we use the Einstein–Weyl equations for a Riemannian submersion in [13] to find when the symmetrised Ricci tensor of the Einstein–Weyl structure, defined by g and its Riemannian connection, is zero. To obtain a Riemannian submersion consider $\hat{g} = g/(\mu\,r^{2\lambda/\mu})$. The metric connection of g is now regarded as a Weyl connection and

$$\hat{\omega} = d\log\left(\frac{1}{\mu\,r^{2\lambda/\mu}}\right) = -\frac{\lambda}{\mu}\frac{dr^2}{r^2}.$$

Now we may write

$$\hat{g} = \mathrm{Re}(\theta_1^2 + \theta_2^2 + \bar{\zeta}^t\otimes\zeta),$$

where

$$\theta_1 = \frac{1}{2\sqrt{\mu}}\left(\frac{1}{z}\alpha + \frac{1}{\bar{z}}\bar{\alpha}\right) \quad\text{and}\quad \theta_2 = \frac{i}{2\sqrt{\mu}}\left(\frac{1}{z}\alpha - \frac{1}{\bar{z}}\bar{\alpha}\right).$$

Thus $\hat{\omega} = -2\lambda\theta_1/\sqrt{\mu}$ and a local computation shows that $\operatorname{div}\hat{\omega} = 0$. Note that

$$d\left(\frac{1}{z}\alpha\right) = -d\omega_- = -\lambda\bar{\zeta}^t\wedge\zeta,$$

so $d\theta_1 = 0$ and $d\theta_2 = -i\mu^{-1/2}\lambda\bar{\zeta}^t\wedge\zeta$. If s is to be a function such that the symmetrised Ricci tensor is $s\hat{g}$, then the Einstein–Weyl equations for a 2-torus bundle in [13] reduce to

$$2n\frac{\lambda^2}{\mu} = \frac{1}{2}s + 2n\frac{\lambda^2}{\mu},$$
$$q - \frac{\lambda^2}{2\mu} = \frac{1}{2}s + 2n\frac{\lambda^2}{\mu}$$

where q is the constant such that $c_1(N) = 2\pi q[\Omega_N]$, where Ω_N is the Kähler form of N. Thus, λ determines a unique solution for μ and the scalar curvature of the resulting Einstein metric must be zero. $\square$

Note that this metric is a degenerate case of a family of Ricci-flat metrics constructed by Calabi [5] (see also [4]). Also note that one may avoid the local computation for div $\hat{\omega}$ by retaining this term in the computations and then observing that the quotient of the canonical bundle by the fibrewise action of $\mathbb{Z}$ is a compact Hermitian–Einstein–Weyl manifold and so, by Proposition 3.2, must be Ricci-flat.

Theorem 4.2. *Every strongly regular compact Hermitian–Einstein–Weyl manifold M of dimension $2n \geqslant 6$ which is not globally conformal to Einstein arises as a fibration over a Kähler–Einstein $(2n-2)$-orbifold N of positive scalar curvature. If N is a manifold, then M is obtained as a discrete quotient of the Ricci-flat Kähler structure on a principal $\mathbb{C}^*$-bundle associated to a maximal root of the canonical bundle of N.*

Proof. It only remains to prove the converse as we have already seen that M is a flat circle bundle over P, which in turn is a circle bundle over a Kähler–Einstein manifold N of positive scalar curvature and the curvature of P is proportional to the Kähler form of N.

$$\begin{tikzcd} M \arrow[rr] \arrow[dr] & & P \arrow[dl] \\ & N & \end{tikzcd}$$

Now by a result of Kobayashi [10], N is simply connected and P must be the circle bundle associated to some rational power of the canonical bundle. Also $\pi_1(P)$ is finite, since P carries an Einstein metric of positive scalar curvature, and passing to its universal cover we may assume that it is simply connected and that M is the trivial circle bundle over P. In this case the universal cover of M is $L \setminus 0$ where L is a maximal root of the canonical bundle. Let g be a representative of the conformal class of M in which the 1-form ω is parallel. Then on $L \setminus 0$, $\omega = df$ and $\hat{g} = e^{-f}g$ is a Ricci-flat Kähler metric. Also

$$\begin{aligned} \omega^\sharp f &= df(\omega^\sharp) = \|\omega\|^2, \\ (I\omega^\sharp)f &= df(I\omega^\sharp) = g(\omega^\sharp, I\omega^\sharp) = 0, \end{aligned}$$

so the $\mathbb{C}^*$-action is by homotheties of $\hat{g}$. This implies that the metric $\hat{g}$ on $L \setminus 0$ is of the form considered in the previous proposition. $\square$

Note that the inverse construction can be used to produce regular Hermitian–Einstein–Weyl structures which are not strongly regular by using a twisted $\mathbb{Z}$-action as in the Remark 2 of Section 3.

Corollary 4.3. *The Weyl connection on a strongly regular compact Hermitian–Einstein–Weyl space of dimension at least six is incomplete.*

Proof. In the notation above, completeness of the Weyl connection is equivalent to completeness of the metric connection for the Kähler scalar flat metric on $L\backslash 0$. However, when restricted to radial directions this metric is of the form $r^a dr^2$, for some constant a, which is an incomplete metric on $(0,\infty)$. □

The above theorem leads to a quotient construction for Hermitian–Weyl manifolds. Suppose a compact Lie group G acts on M preserving the complex structure and conformal class and commuting with the actions generated by $\omega^\sharp$ and $I\omega^\sharp$ and such that for each vector field X generated by the action of G we have $L_X\omega = 0$. This action descends to $N = M/\mathcal{E}$ and we may construct a Kähler moment map $\mu_N\colon N \to \mathfrak{g}^*$. Let μ_M be the pull-back of μ_N to M. Futaki [6] showed that the Einstein property is preserved by the Kähler quotient if each vector field generated by the action of G on N has constant length on the level set of μ_N. If this is satisfied and G acts freely on $\mu_M^{-1}(0)$ we may consider the manifold $M//G = \mu_M^{-1}(0)/G$. This inherits a complex structure and a conformal structure from M and one may now see that it is Hermitian–Einstein–Weyl since it is constructed as a bundle over the Kähler–Einstein manifold $\mu_N^{-1}(0)/G$.

We may also obtain a classification of the six-dimensional, compact, strongly regular Hermitian–Einstein–Weyl manifolds since it is known that a compact Kähler–Einstein four-manifold of positive scalar curvature must be either $\mathbb{CP}(1)\times\mathbb{CP}(1)$, $\mathbb{CP}(2)$ or $\mathbb{CP}(2)$ blown-up at between 3 and 8 points in general position (see [16, 17]).

5. HyperHermitian–Weyl manifolds

Recall that a manifold is *hypercomplex* if it admits three complex structures I, J and K such that

$$I^2 = J^2 = -1 \quad \text{and} \quad IJ = K = -JI. \tag{5.1}$$

A metric g is *hyperHermitian* (respectively *hyperKähler*) if it is Hermitian (Kähler) with respect to each of these complex structures. If $(M,[g],D)$ is a conformal manifold for which g is hyperHermitian and where D is a compatible Weyl connection preserving the complex structures, then we will call M a *hyperHermitian–Weyl* space.

These notions are a special case of the following. Let M be a manifold with a subbundle $\mathcal{G}$ of $\operatorname{End} TM$, the endomorphism bundle of the tangent bundle of M, such that locally $\mathcal{G}$ has a basis $\{I,J,K\}$ satisfying (5.1). A metric g is said to be *quaternion-Hermitian* if for each element A of such a local basis, $g(AX,AY) = g(X,Y)$. Locally one has 2-forms Ω_A defined by $\Omega_A(X,Y) = g(X,AY)$ and one may define a global 4-form by the local formula

$$\Omega = \Omega_I\wedge\Omega_I + \Omega_J\wedge\Omega_J + \Omega_K\wedge\Omega_K.$$

If M is at least eight-dimensional, then one says that M is *quaternionic Kähler* if Ω is parallel with respect to the Levi-Civita connection of g. On a quaternion-Hermitian manifold this is equivalent to requiring that the Levi-Civita connection preserves the bundle $\mathcal{G}$. Accordingly, a conformal manifold $(M,[g],\mathcal{G},D)$ will be called *quaternion-Hermitian–Weyl* if g is quaternion-Hermitian and D is a Weyl connection preserving $\mathcal{G}$. Note that in general the local almost complex structures I, J, K will not be integrable.

Proposition 5.1. (1) *Let M be a manifold of dimension $4n$ and if $n = 1$ assume that M is compact. Then M is hyperHermitian–Weyl if and only if M is locally conformal hyperKähler. In particular, any such hyperHermitian manifold is Einstein–Weyl.*

(2) *A manifold of dimension at least eight is quaternion-Hermitian–Weyl if and only if it is locally conformal to quaternionic Kähler. So any such manifold is Einstein–Weyl.*

Proof. (1) After the discussion in section 2 it only remains to show that on a compact hyperHermitian four-manifold the 1-forms given by $Dg = \omega \otimes g$ are necessarily closed. Let Ω_I be the fundamental 2-form associated to I and g. Then Ω_I is anti-self-dual and $d\omega \wedge \Omega_I = 0$. Since this also holds for J and K, this implies that $d\omega$ is self-dual $d\omega = *d\omega$. Now, since M is compact,

$$\int_M \|d\omega\|^2 = \int_M d\omega \wedge *d\omega = \int_M d\omega \wedge d\omega = \int_M d(\omega \wedge d\omega) = 0,$$

so $d\omega$ is identically zero.

The last assertion follows from the fact that a hyperKähler manifold is Ricci-flat [2].

(2) From [14] we have that a quaternion Hermitian manifold of dimension at least 12 is quaternionic Kähler if and only if the 4-form Ω is closed. Whereas in [15] it was shown that a quaternion Hermitian 8-manifold is quaternionic Kähler if and only if $d\Omega = 0$ and whenever we have a local basis $\{I, J, K\}$ for $\mathcal{G}$, $d\Omega_I = a \wedge \Omega_I + b \wedge \Omega_J + c \wedge \Omega_K$ for some 1-forms a, b, c.

On a quaternion-Hermitian–Weyl manifold, D preserves $\mathcal{G}$ and we choose a local basis I, J, K satisfying (5.1), so locally there are 1-forms α, β, γ such that

$$\begin{aligned}
DI &= \alpha \otimes J - \beta \otimes K,\\
DJ &= -\alpha \otimes I + \gamma \otimes K,\\
DK &= \beta \otimes I - \gamma \otimes J.
\end{aligned}$$

Thus $d\Omega_I = \omega \wedge \Omega_I - \alpha \wedge \Omega_J + \beta \wedge \Omega_K$ and $d\Omega = \omega \wedge \Omega$. The arguments of section 2 now show that M is locally conformal to quaternionic Kähler if its dimension is not four.

The final assertion follows from the fact that quaternionic Kähler manifolds are Einstein [3, 1]. □

On a hyperHermitian–Weyl manifold, choose a representative metric for which ω is parallel and let $\mathcal{E}$ be the foliation generated by $\omega^\sharp$, $I\omega^\sharp$, $J\omega^\sharp$ and $K\omega^\sharp$.

Theorem 5.2. *Every compact hyperHermitian–Weyl manifold of dimension $4n$ which is not hyperKähler and for which the leaves of the foliation $\mathcal{E}$ are closed and 4-dimensional arises as a fibration over a quaternionic Kähler $(4n-4)$-orbifold of positive scalar curvature.*

Proof. Define $\theta_I = -\omega \circ I$, etc. Then $\Omega_J = -\omega \wedge \theta_J + \theta_I \wedge \theta_K + \Psi_J$ for some 2-form Ψ_J which is zero on the quaternionic span of $\omega^\sharp$. Now

$$\begin{aligned} L_{I\omega^\sharp}\Omega_J &= d(I\omega^\sharp \lrcorner \Omega_J) + I\omega^\sharp \lrcorner d\Omega_J \\ &= -d(\omega^\sharp \lrcorner \Omega_K) + I\omega^\sharp \lrcorner (\omega \wedge \Omega_J) \\ &= -d(\omega^\sharp \lrcorner \Omega_K) + \omega \wedge (\omega^\sharp \lrcorner \Omega_K) \\ &= -L_{\omega^\sharp}\Omega_K + \Omega_K \\ &= \Omega_K. \end{aligned}$$

Thus the quaternionic span of $\omega^\sharp$ generates an action of $U(1) \times Sp(1)$ on M permuting the complex structures. On the universal cover $\tilde{M}$ of M the structure is globally conformal to hyperKähler and $\omega = df$. Now

$$\begin{aligned} \omega^\sharp f &= df(\omega^\sharp) = \|\omega\|^2, \\ (I\omega^\sharp)f &= df(I\omega^\sharp) = g(\omega^\sharp, I\omega^\sharp) = 0 \end{aligned}$$

so the action lifts to a homothetic action of the hyperKähler structure. From [15] it follows that the quotient B of M by this group action is a quaternionic Kähler orbifold of positive scalar curvature and that the hyperKähler metric on $\tilde{M}$ is locally isometric to the hyperKähler structure on a bundle $\mathcal{U}(B)$ over B constructed as follows. Let F be the bundle of frames of B compatible with the quaternionic Kähler structure. This is a principal bundle with structure group $Sp(n)\,Sp(1) = (Sp(n) \times Sp(1))/\{\pm 1\}$. The bundle $\mathcal{U}(B)$ is now defined to be $(F \times (\mathbb{H}^*/\{\pm 1\}))/(Sp(n)\,Sp(1))$, where $[A, q] \in Sp(n)\,Sp(1)$ acts taking $(u, [\xi])$ to $(u \cdot [A, q], [\xi q])$. Note that the hyperKähler metric on $\mathcal{U}(B)$ admits a homothetic action of $\mathbb{Z}$ and that the quotient is a compact hyperHermitian manifold if B is compact. $\square$

Note that the above proof also shows that the local hyperKähler metrics admit hyperKähler potentials, that is functions which are Kähler potentials for each of the Kähler structures simultaneously. In this case we may write the local hyperKähler metric in terms of its Levi-Civita connection as ∇df for some function f [15].

If we regard a hyperHermitian–Weyl manifold as just a Hermitian–Einstein–Weyl manifold, then the resulting Kähler–Einstein manifold obtained in section 3 is just the twistor space of the quaternionic Kähler manifold B associated to the hyperHermitian–Weyl structure. This gives the following diagram:

$$\begin{tikzcd} \mathcal{U}(B) \arrow[r, "\mathbb{Z}"] \arrow[dr, "\mathbb{H}^*/\mathbb{Z}"'] & M \arrow[r, "S^1"] \arrow[d, "F"'] \arrow[dr] & P \arrow[d, "S^1"] \\ & B & Z \arrow[l, "S^2"] \end{tikzcd}$$

where M is a compact hyperHermitian–Weyl manifold, B is quaternionic Kähler with positive scalar curvature, Z is the twistor space of $B, \mathcal{U}(B)$ is the hyperKähler manifold associated to B, the fibres F of $M \to B$ are Hopf manifolds $S^1 \times S^3$ and the circle bundle P over Z carries an Einstein metric of positive scalar curvature. This generalises the following picture:

$$\begin{array}{ccccc} \mathbb{H}^{n+1}\setminus\{0\} & \longrightarrow & S^{4n+3}\times S^1 & \longrightarrow & S^{4n+3} \\ & \searrow & \downarrow & \searrow & \downarrow \\ & & \mathbb{HP}(n) & \longleftarrow & \mathbb{CP}(2n+1) \end{array}$$

Note that one can again obtain examples where the quaternionic Kähler quotient B is an orbifold by starting with $M = S^1 \times S^{4n+3}/\mathbb{Z}_p$ as described in remark 4 of section 3.

Proposition 5.3. *A compact quaternion-Hermitian-Weyl manifold which is not quaternionic Kähler is foliated by four-dimensional self-dual totally geodesic submanifolds which are locally conformal to Einstein.*

Proof. Let g be a representative of the conformal class for which $d^*\omega = 0$. We have already shown that locally there is another representative $\hat{g}$ which is quaternionic Kähler. Now

$$\hat{\nabla}\omega^\sharp = D\omega^\sharp = \nabla\omega^\sharp + a(\omega^\sharp) = -\tfrac{1}{2}\,\mathrm{Id}\,.$$

Let X be the vector field $-2\omega^\sharp$, then $\hat{\nabla}X = \mathrm{Id}$. If $\{I, J, K\}$ is a local basis of $\mathcal{G}$ let $\mathcal{D}$ be the distribution spanned by X, IX, JX and KX. Note that this is a well-defined distribution on the whole of M. Define local 1-forms α, β, γ as before, then

$$\begin{aligned} &D_{IX}X = IX, \\ &D_X IX = (D_X I)X - ID_X X = \alpha(X)JX - \beta(X)KX - IX, \\ &D_{IX}JX = (D_{IX}J)X - JD_{IX}X = -\alpha(IX)IX + \gamma(IX)KX + KX \end{aligned}$$

so $[X, IX] = D_X IX - D_{IX}X \in \mathcal{D}$ and $[IX, JX] \in \mathcal{D}$. Thus $\mathcal{D}$ is integrable and the integral manifolds are totally geodesic. Since $\mathcal{D}$ is an $\mathbb{H}$-submodule of TM, a result of Marchiafava [11] implies that the restriction of $\hat{g}$ to $\mathcal{D}$ is a self-dual Einstein metric. □

The equation $\hat{\nabla}\omega = -\frac{1}{2}\,\mathrm{Id}$ shows that locally we may find a function f such that the quaternionic Kähler metric satisfies

$$\hat{g} = \hat{\nabla}df.$$

We will call such an f a *quaternionic Kähler potential* in analogy with the hyperKähler case. This should be regarded as a type of moment map not for a group action but for the foliation generated by IX, JX and KX.

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