CONFORMAL TRANSFORMATIONS OF COMPACT SELF-DUAL MANIFOLDS

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We prove that when the dimension of the group of conformal transformations of a compact self-dual manifold is at least three, the conformal class contains either a metric with positive constant scalar curvature or a metric with zero scalar curvature. This result is combined with a topological classification of 4-manifolds to provide a complete geometrical classification of the compact self-dual manifolds whose symmetry group is at least three-dimensional.

1. Introduction

A four-dimensional oriented Riemannian manifold (X, g) is called self-dual if the anti-self-dual part of the Weyl tensor of the metric g vanishes [1]. Since both the Weyl tensor and the Hodge $*$-operator on the 2-forms of a four-dimensional Riemannian manifold are conformally invariant, the conformal class of metrics containing such a metric g consists entirely of self-dual metrics. The symmetry group of a self-dual metric may therefore naturally be said to be the group of orientation preserving conformal transformations.

The most striking result in the subject of self-duality is a recent theorem of Taubes [40] to the effect that for each compact oriented four-dimensional manifold M, there are self-dual conformal classes on connected-sums of M with sufficiently many copies of the complex projective plane $\mathbf{CP}^2$. In particular, self-dual conformal structures on compact manifolds exist in abundance. However, in order to produce specific examples of self-dual manifolds, one typically appeals to some *algebraic* means [16], [17], [18], [35]. A very successful such method is LeBrun's *hyperbolic Ansatz* for the construction of self-dual manifolds with S^1 symmetry [11], [19]. For example, this method can be applied to construct self-dual metrics on the connected-sums of many copies of the complex projective plane

$$n\mathbf{CP}^2 := \underbrace{\mathbf{CP}^2 \# \dots \# \mathbf{CP}^2}_{n \text{ times}}$$

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and on the connected-sums of many copies of the Hopf manifold with the complex projective plane

$$m(S^1 \times S^3) \# n\mathbf{CP}^2 := \underbrace{(S^1 \times S^3) \# \ldots \# (S^1 \times S^2)}_{m \text{ times}} \# \underbrace{\mathbf{CP}^2 \# \ldots \# \mathbf{CP}^2}_{n \text{ times}}.$$

The group of conformal transformations of these self-dual metrics are typically one-dimensional, but on $n\mathbf{CP}^2$, some of these metrics have a two-dimensional torus as a symmetry group. With these examples of self-dual manifolds with non-trivial group of conformal transformations in mind, a natural question is, 'What are the self-dual manifolds whose group of conformal transformations is at least *three*-dimensional?' In this article, we shall give a complete geometrical classification for such self-dual manifolds.

To state our results, recall that every conformal class on any compact manifold contains a metric of constant scalar curvature [38]. The sign of the scalar curvature of this metric is an invariant of the conformal class, called its *type*. The major technical result towards our geometrical classification is the following:

Theorem 1.1. *If the dimension of the group of conformal transformations of a compact self-dual manifold (X, g) is at least three, the conformal class of g has non-negative type.*

Due to Theorem 1.1, the next two theorems provide a complete geometrical classification of compact self-dual manifolds with large group of conformal transformations.

Theorem 1.2. *If the group of conformal transformations of a compact self-dual manifold (X, g) is at least three-dimensional, and if the scalar curvature of g is zero, then (X, g) is conformally equivalent to one of the following:*

(1) *the flat torus;*
(2) *the conformally flat $S^2 \times \Sigma_g$ or its finite quotient, where Σ_g is a Riemann surface of genus $g \geq 2$ with a metric of constant sectional curvature -1.*

Theorem 1.3. *If the group of conformal transformations of a compact self-dual manifold (X, g) is at least three-dimensional, and if the scalar curvature of g is positive, then (X, g) is conformally equivalent to one of the following:*

(1) *the Euclidean 4-sphere, S^4;*
(2) *the complex projective plane $\mathbf{CP}^2$ with the Fubini-Study metric;*
(3) *finitely covered by the Riemannian product of the Euclidean sphere metrics on $S^1 \times S^3$;*
(4) *the conformal structure $C(r, p)$ on the connected-sums $r(S^1 \times S^3) \# p(S^1 \times \mathbf{RP}^3)$.*

In this result, we have introduced the following terminology:

Definition 1.4. The conformal structures $C(r, p)$ on the connected-sum of r-copies of $S^1 \times S^3$ and p-copies of $S^1 \times \mathbf{RP}^3$ are conformal compactifications of Riemannian product $S^2 \times \Sigma$, where S^2 is the 2-dimensional sphere with constant sectional curvature 1 and Σ is an open Riemann surface with constant sectional curvature -1. Here

Σ has genus g and $(r - g + 1) + p$ many boundary components, where $r \geq g$. The compactification is obtained by adding p-many copies of $\mathbf{RP}^2 \times S^1$ and $(r - g + 1)$-many copies of S^1 onto the boundary of Σ [27].

Example. $S^1 \times S^3$ can be obtained by adding two circles onto the product of the 2-sphere with an annulus. $S^1 \times \mathbf{RP}^3$ can be obtained by adding a circle and a copy of $\mathbf{RP}^2 \times S^1$ onto the product of the 2-sphere with an annulus.

The conformal classes $C(r, p)$ have non-trivial moduli because the hyperbolic metrics on the open Riemann surface Σ have non-trivial moduli. The manifold $r(S^1 \times S^3) \# p(S^1 \times \mathbf{RP}^3)$ has some other interesting features: Recall that the signature of a compact 4-dimensional manifold is equal to a constant multiple of the difference of the L^2-norm of the self-dual Weyl tensor and the anti-self-dual Weyl tensor [3]. As the signature of the manifold $r(S^1 \times S^3) \# p(S^1 \times \mathbf{RP}^3)$ is equal to zero, any self-dual conformal class on this manifold is conformally flat. It is known that there are many different conformally flat structures on this manifold. Moreover, it is possible to vary the conformal classes continuously on the 'moduli' space of conformally flat structures so that the conformal classes vary from positive type to negative type [11], [25], [39]. Yet, topologically, this manifold always admits the group SO(3) as a group of diffeomorphisms.

As far as the group G is concerned, we can and we shall assume that it is either SU(2), SO(3) or the 3-dimensional torus T^3. The reason is due to Obata's theorem stating that the identity component of the group of conformal transformations of a compact manifold is non-compact only when the conformal structure is globally conformally equivalent to the standard sphere [12], [14], [26]. Leaving this special case out, we assume that G is a compact Lie group. Then it is well-known that when G is non-abelian, it must contain either SU(2) or SO(3) as a subgroup. Therefore, for our purpose of classification, it is enough to assume that G is one of the SU(2), SO(3) or T^3.

Putting the geometry aside, Theorem 1.2 and Theorem 1.3 are not new at all. Indeed, the topological problem of classifying 4-manifolds with large compact group of diffeomorphisms has a long history. It began with the work of Moster in 1957 [23], [24], through Pak [30], Orlik [27] and Melvin [21]. A complete topological classification was obtained by Melvin and Parker in 1986 [22]. Their results, along with Theorem 1.1 are the two stepping stones towards our geometrical classification.

When one compares Melvin and Parker's list of topological classification with our lists in Theorem 1.2 and Theorem 1.3, the items missing in our lists are those manifolds finitely covered by $S^2 \times S^2$ or by $\mathbf{CP}^2 \# \overline{\mathbf{CP}}^2$. Note that the signatures of these manifolds are equal to zero. If these manifolds were to have any self-dual conformal class, the conformal class would have been conformally flat; but, due to Kuiper's theorem [13], [14], a simply-connected manifold other than S^4 cannot be conformally flat. Therefore, an alternative presentation of our result is the following:

Theorem 1.5. *Supposed that X is a compact oriented four-dimensional manifold. Suppose also that there is a three-dimensional compact connected Lie group G acting on*

X effectively as a group of diffeomorphisms. Then unless X is finitely covered by $S^2 \times S^2$ or $\mathbf{CP}^2 \# \overline{\mathbf{CP}}^2$, there is a self-dual conformal class of non-negative type on X such that the group G acts as a group of conformal transformations.

Remark 1.6. Joyce recently generalized LeBrun's approach [19] to produce a large collection of self-dual metrics with 2-torus symmetry on connected-sums of the complex projective plane [10]. Using a very different approach, namely an equivariant version of the Donaldson-Friedman's programme [5], the author and Pedersen have also produced similar metrics. It may be shown [31] that all these metrics are of positive type.

By constrast, there are plenty of negative type self-dual conformal classes on compact manifolds with the group U(1) as its group of conformal transformations [11], [25]. It would therefore be very interesting to investigate the situation when a self-dual conformal class admits a 2-dimensional torus as its group of conformal transformations. In view of the incompleteness of the topological classification in this case [28], [29], one should at least answer a question reflecting the result of Theorem 1.1:

Question 1.7. When there is a 2-torus acting on a compact self-dual manifold as a group of orientation preserving conformal transformations, is the self-dual conformal class necessarily non-negative type?

Either an affirmative answer by proof or a counter example by construction will be very useful for the understanding of the conformal geometry of a self-dual conformal class.

2. Twistor Theory and Examples

A twistor space Z is the total space of the bundle of unit anti-self-dual 2-forms of an oriented four-dimensional Riemannian manifold (X, g). Using the metric g to define a linear isomorphism between the space of tangent vectors and the space of 1-forms, we identify the twistor space to be the space of almost complex structures on the manifold X such that the canonical orientation of the almost complex structures are opposite to the given one on X. Tautologically, the twistor space has an almost complex structure. A fundamental observation in twistor theory is that the naturally defined almost complex structure on the twistor space is integrable if and only if the metric g on the manifold X is self-dual [1], [3]. Moreover, the fibres of the projection from Z onto X are holomorphic rational curves with holomorphic normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ and they will be referred to as the real twistor lines [3], [9].

It is well-known that the twistor space completely encodes all the conformal geometry of the Riemannian manifold (X, g) [1], [8]. For example, if φ is an orientation preserving conformal transformation, it can be lifted to be a *holomorphic* transformation of Z [3]. The lifted action $\hat{\varphi}$ on Z is defined by

$$\hat{\varphi}(z) = \frac{(\varphi^{-1})^* z}{\|(\varphi^{-1})^* z\|},$$

where z is considered as an anti-self-dual 2-form.

When Y is a vector field on X generated by a 1-parameter group of conformal transformations φ_t and V is the corresponding holomorphic vector field on Z generated by $\hat{\varphi}_t$, then as real tangent vectors,

$$dp(\text{the real part of } V) = Y. \tag{2.1}$$

We shall use $\hat{G}$ to denote the group of holomorphic transformations on Z generated by the G-action on the manifold X.

Another feature of the holomorphic vector field V is that it is real with respect to the real structure of the twistor space [9]. The real structure is the anti-holomorphic involution of the twistor space defined by the fibre-wise anti-podal map $z \to -z$. In terms of this real structure, the real part of the space of holomorphic vector fields on a twistor space is isomorphic to the space of conformal Killing vector fields on the manifold X [8]. We shall illustrate this aspect of twistor correspondence by a very useful example.

Example 2.2. The twistor space of the Euclidean 4-sphere S^4 is the complex projective 3-space $\mathbf{CP}^3$ [3]. Let $[z_0, z_1, z_2, z_3]$ be the homogeneous coordinates on $\mathbf{CP}^3$ and let S^4 be considered as a projective quaternionic space with homogeneous coordinates $[q_0, q_1]$, then the twistor fibration is the map

$$[z_0, z_1, z_2, z_3] \to [z_0 + z_1 j, z_2 + z_3 j].$$

An action of the group $Sp(1)$ on the manifold S^4 can be defined by the map

$$[z_0 + z_1 j, z_2 + z_3 j] \to [(z_0 + z_1 j)(a + bj), z_2 + z_3 j],$$

where $|a|^2 + |b|^2 = 1$. This action is lifted onto the twistor space to be

$$[z_0, z_1, z_2, z_3] \to [z_0 a - z_1 \bar{b}, z_0 b + z_1 \bar{a}, z_2, z_3].$$

When the matrices

$$A_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad A_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.3}$$

are used as a basis for the Lie algebra $sp(1) \cong su(2)$, the holomorphic vector fields generated by them are:

$$V_1 = i\left(z_0 \frac{\partial}{\partial z_0} - z_1 \frac{\partial}{\partial z_1}\right);$$

$$V_2 = i\left(z_1 \frac{\partial}{\partial z_0} + z_0 \frac{\partial}{\partial z_1}\right);$$

$$V_3 = -z_1 \frac{\partial}{\partial z_0} + z_0 \frac{\partial}{\partial z_1}.$$

As the twistor space of the Euclidean 4-sphere, the complex projective 3-space has a real structure defined by the anti-holomorphic involution:

$$\sigma[z_0, z_1, z_2, z_3] = [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2].$$

Then it is easy to check that the $Sp(1)$-action on $\mathbf{CP}^3$ described above is real and that

$$d\sigma(V_i) = \bar{V}_i, \qquad \text{for } i = 1, 2, 3.$$

It means that the V_i's are real holomorphic vector fields.

If one removes the point $[0, 1]$ from the sphere, one can conformally identify $S^4 \backslash [0, 1]$ to the space of the quaternions $\mathbf{H}$. As real vector spaces, there is a natural linear isomorphism: $\mathbf{H} \cong \mathbf{R}^4$. Therefore, with respect to the standard coordinate chart (x_0, x_1, x_2, x_3) on $\mathbf{R}^4$, the vector fields Y_1, Y_2 and Y_3 on $S^4 \backslash [0, 1]$ can be expressed as follows:

$$Y_1 = -x_1 \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3};$$

$$Y_2 = -x_3 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_0 \frac{\partial}{\partial x_3};$$

$$Y_3 = -x_2 \frac{\partial}{\partial x_0} - x_3 \frac{\partial}{\partial x_1} + x_0 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}.$$

An interesting aspect of this example is that while the maximum dimension of the $\mathbf{R}$-linear span of Y_1, Y_2 and Y_3 on $\mathbf{R}^4$ is equal to three, the maximum dimension of the $\mathbf{C}$-linear span of V_1, V_2 and V_3 on the twistor space is equal to 2. To be more precise,

$$V_2 \wedge V_3 = 2iz_0 z_1 \frac{\partial}{\partial z_0} \wedge \frac{\partial}{\partial z_1}$$

$$V_3 \wedge V_1 = -i(z_0^2 - z_1^2) \frac{\partial}{\partial z_0} \wedge \frac{\partial}{\partial z_1}$$

$$V_1 \wedge V_2 = -(z_0^2 + z_1^2) \frac{\partial}{\partial z_0} \wedge \frac{\partial}{\partial z_1}.$$

Therefore, away from the real twistor lines defined by $\{z \in \mathbf{CP}^3 : z_0 = z_1 = 0\}$ and $\{z \in \mathbf{CP}^3 : z_2 = z_3 = 0\}$, the distribution of V_1, V_2 and V_3 has constant rank 2.

To capture the set of zeros of the holomorphic tensors fields $V_2 \wedge V_3$, $V_3 \wedge V_1$ and $V_1 \wedge V_2$, we define

$$s_1 := 2iz_0z_1, \qquad s_2 := -i(z_0^2 - z_1^2), \qquad s_3 := -(z_0^2 + z_1^2). \tag{2.4}$$

For further application of this example, we note that these homogeneous polynomials are quadratic and are reducible. Invariantly speaking, s_1, s_2 and s_3 are holomorphic sections of the degree 2 bundle on $\mathbf{CP}^3$. Their divisors are reducible.

3. Proof of Theorem 1.1

When the group G of conformal transformations of a compact Riemannian manifold (X,g) is non-abelian, it contains either SU(2) or SO(3) as a subgroup. In this case, we take the matrices A_1, A_2 and A_3 in (2.3) as a basis for the Lie algebra. When G is abelian, we assume that it is the 3-torus group T^3. Then A_1, A_2 and A_3 will denote three commuting vectors generating the algebra of G. In any case, V_1, V_2 and V_3 will be the corresponding holomorphic vector fields on the twistor space Z. In order to prove Theorem 1.1, we shall investigate the holomorphic distribution determined by this algebra of holomorphic vector fields.

Lemma 3.1. *The distribution determined by any pair of the vector fields V_1, V_2 and V_3 is rank* 2.

Proof. Let us consider V_1 and V_2. Let $\mathscr{S}$ denote the image sheaf of the morphism on the twistor space

$$\mathcal{O} \oplus \mathcal{O} \to \mathscr{T}\mathscr{Z}$$

defined by

$$(f_1, f_2) \to f_1 V_1 + f_2 V_2 .$$

Obviously, the rank of $\mathscr{S}$ is at least 1. Since the image sheaf is a subsheaf of the tangent sheaf $\mathscr{T}\mathscr{Z}$, $\mathscr{S}$ is coherent and torsion-free [7]. If its rank is equal to 1, then its bi-dual sheaf $\mathscr{S}^{**}$ is locally-free. Hence, via the natural monomorphism of a torsion-free sheaf into its bi-dual sheaf

$$\mathscr{S} \to \mathscr{S}^{**},$$

we may consider V_1 and V_2 as holomorphic sections of a complex line bundle. Therefore, their ratio is a meromorphic function. This meromorphic function cannot be a constant because V_1 and V_2 are linearly independent over $\mathbf{C}$. It follows that V_1 vanishes on the divisor of zeroes of the non-constant meromorphic function V_1/V_2. So does the real part of V_1. Due to the twistor projection (2.1), Y_1 would have vanished identically on the manifold X. This is impossible due to the effectiveness of the group action on X.

Let $\mathscr{D}$ denote the image sheaf of the morphism on the twistor space

$$\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \to \mathscr{T}\mathscr{Z}$$

defined by

$$(f_1, f_2, f_3) \to f_1 V_1 + f_2 V_2 + f_3 V_3 .$$

According to Lemma 3.1, the rank of $\mathscr{D}$ is at least 2.

When the rank of $\mathscr{D}$ is equal to 2, V_1, V_2 and V_3 are **C**-linearly dependent everywhere. In this case, let **E** be the determinant bundle of $\mathscr{D}$. It is defined by the isomorphism [7]

$$(\wedge^2 \mathscr{D})^{**} \cong \mathcal{O}(\mathbf{E}).$$

Via the natural monomorphism:

$$(\wedge^2 \mathscr{D}) \to (\wedge^2 \mathscr{D})^{**},$$

we consider

$$s_1 := V_2 \wedge V_3 , \qquad s_2 := V_3 \wedge V_1 , \qquad s_3 := V_1 \wedge V_2 \tag{3.2}$$

as holomorphic sections of the line bundle **E**. Since the distribution is rank 2, not all of the s_i's is identically zero. Therefore, the real line bundle **E** has non-trivial holomorphic section.

Lemma 3.3. *When the distribution sheaf $\mathscr{D}$ is rank 2, its determinant line bundle has at least two linearly independent sections.*

Proof. If the group G is either SU(2) or SO(3), then the linear span of s_1, s_2 and s_3 in the vector space $H^0(Z, \mathcal{O}(\mathbf{E}))$ is the complexification of the adjoint representation of the group G and hence is at least three-dimensional.

When the group G is a 3-torus, we assume that the section s_1 is not identically zero. Suppose that there is a constant c such that $s_1 = cs_2$, i.e.

$$(V_2 + cV_1) \wedge V_3 \equiv 0. \tag{3.4}$$

As the vector fields V_i's are real, so are the sections s_1, s_2. Therefore, the constant c is real. Define $V_2^{\sim} := V_2 + cV_1$, then $\{V_1, V_2^{\sim}, V_3\}$ forms a new basis for the algebra of real holomorphic vector fields generated by A_1, A_2 and A_3. Then the identity (3.4) is a contradiction to Lemma 3.1. Therefore, the three sections s_1, s_2 and s_3 are mutually linearly independent.

Proposition 3.5. *When the distribution sheaf $\mathscr{D}$ is rank 2, there is a real holomorphic line bundle* **F** *with $c_1(\mathbf{F}) = 0$ such that the determinant line bundle* **E** *is isomorphic to* $\mathbf{K}^{-1/2}\mathbf{F}$.

Proof. The reality of the V_i's implies that $\mathscr{D}$ and hence $\mathcal{O}(\mathbf{E})$ is real. Therefore, there is a real holomorphic line bundle $\mathbf{F}$ with $c_1(\mathbf{F}) = 0$ and an integer n such that [36]

$$\mathbf{E} = \mathbf{K}^{-n/2}\mathbf{F}.$$

Due to Lemma 3.3, $\mathbf{E}$ has at least two linearly independent non-trivial sections. It follows that n is strictly positive because the number $2n$ is the degree of the restriction of the bundle $\mathbf{E}$ onto any real twistor line.

As $\mathscr{D}$ is a coherent torsion-free sheaf, the codimension of its set of singularity is at least 2 [7]. It follows that there is a real twistor line L contained in a neighbourhood on which $\mathscr{D}$ is locally free. On this neighbourhood, $\mathscr{D}$ is considered as a subbundle of the tangent bundle and hence $\mathbf{E}$ is a subbundle of $\bigwedge^2 \mathbf{T}Z$. To calculate the degree of $\mathbf{E}$, we restrict $\mathbf{E}$ onto L to be a subbundle of $\bigwedge^2 \mathbf{T}Z_{|L}$. On the real twistor line L, $\mathbf{T}Z_{|L}$ splits into the direct-sum of the tangent bundle of L, $\mathbf{T}L$, and the normal bundle $\mathbf{N}$. Then

$$\begin{aligned}\textstyle\bigwedge^2 \mathbf{T}Z_{|L} &\cong \textstyle\bigwedge^2 (\mathbf{T}L \oplus \mathbf{N}) \\ &\cong (\mathbf{T}L \otimes \mathbf{N}) \oplus \textstyle\bigwedge^2 \mathbf{N} \\ &\cong \mathcal{O}(3) \oplus \mathcal{O}(3) \oplus \mathcal{O}(2).\end{aligned}$$

As $\mathbf{E}_{|L}$ is a subbundle of $\bigwedge^2 \mathbf{T}Z_{|L}$ with positive degree $2n$, and the only line subbundle with positive even degree in $\mathcal{O}(3) \oplus \mathcal{O}(3) \oplus \mathcal{O}(2)$ is $\mathcal{O}(2)$, therefore, $n = 1$.

Proof of Theorem 1.1. When the distribution sheaf $\mathscr{D}$ has rank 2, its determinant line bundle $\mathbf{K}^{-1/2}\mathbf{F}$ has non-trivial sections according to Lemma 3.3 and Proposition 3.5. When the distribution sheaf has rank 3, the anti-canonical bundle $\mathbf{K}^{-1}$ has a non-trivial section, namely,

$$w := V_1 \wedge V_2 \wedge V_3. \tag{3.6}$$

Then our theorem is a direct consequence of Gauduchon's theorem [6], [36] stating that the only holomorphic section of any non-trivial holomorphic line bundle over the twistor space of a negative type self-dual conformal class on a compact manifold is the trivial section.

4. Symmetry of Type Zero Self-dual Metrics

In this section, we shall prove Theorem 1.2 which states that

Theorem 1.2. *If the group of conformal transformations of a compact self-dual manifold (X, g) is at least three-dimensional and if the scalar curvature of g is equal to zero, then (X, g) is conformally equivalent to one of the following*:

(1) *the flat torus*;

(2) *the conformally flat $S^2 \times \Sigma_g$ or its finite quotient, where Σ_g is a Riemann surface of genus $g \geq 2$ with a metric of constant sectional curvature -1.*

As in the proof of Theorem 1.1, we need to analyze two possibilities.

(i) When the section $V_1 \wedge V_2 \wedge V_3$ is identically zero, the rank of $\mathscr{D}$ is equal to 2. As a consequence of Lemma 3.3, the algebraic dimension of the twistor space is positive. By a theorem of Pontecorvo [33], [37], the conformal class of the metric g either contains or is finitely covered by a Ricci-flat metric. As a self-dual Ricci-flat manifold is either covered by a flat torus or a $K3$-surface with the Calabi-Yau metric [3], the presence of conformal Killing vector fields implies that (X, g) is finitely covered by the flat torus.

(ii) Suppose that $w = V_1 \wedge V_2 \wedge V_3$ is a non-trivial section of the anti-canonical bundle $\mathbf{K}^{-1}$ of the twistor space. Let M be an irreducible component of the zero divisor of the section w. According to Pontecorvo [33], the metric g on the manifold X is pulled back by the twistor fibration onto M to be a scalar-flat Kähler metric. Moreover, the section w is $\hat{G}$-invariant, so is the component M. Therefore, the group of holomorphic transformations of M is at least 3-dimensional.

As the group action $\hat{G}$ on M is a complexification of the action of the compact Lie group G, one may conveniently cite [Proposition 3.1, 20] to deduce that M is either covered by a flat torus or is isometric to $S^2 \times \Sigma_g$ with $g \geq 2$. We can give more details than a citation.

When (M, g) is a scalar-flat Kähler surface, the manifold M is subjected to numerous topological constraints. Therefore, with the aids of the Enriques-Kodaira classification of compact complex surfaces [2], it is not hard to classify such complex manifolds. In particular, if M is not minimal, then M is one of the following [4]:

(1) a ruled surface of genus $g \geq 1$; blown-up at least once;
(2) a rational ruled surface blown-up at least nine times;
(3) $\mathbf{CP}^2$ blown-up at least ten times.

Since the compact group G acts on M as a group of holomorphic transformations and such transformations leave any exceptional divisors of blowing-up invariant, the G-action on M descends onto a minimal model S of M as a group of holomorphic transformations and leaves all points of blowing-up fixed. Let f be the blowing-down map from M onto S. Then $df(V_i)$, $i = 1, 2, 3$, are holomorphic vector fields on S vanishing at the points of blowing-up, say $p_1, \ldots, p_k$.

When M is not minimal, the Kähler metric cannot be Ricci-flat because the canonical bundle is non-trivial. Therefore, due to the vanishing of the scalar curvature, the canonical bundle and all its non-zero power do not admit any non-trivial sections [41]. In particular, if the minimal model S of the complex surface M is $\mathbf{CP}^2$, there are at least three of the collection $\{p_1, \ldots, p_k\}$ non-collinear. Therefore, M does not support a three-dimensional group of holomorphic transformations.

In the cases (2) and (3), the minimal model S is a ruled surface over a Riemann surface. When S is blown-up to M, the singular fibres of the ruling are composed of irreducible rational curves of negative self-intersection numbes and hence are invariant under the action of G. As a singular fibre is a connected set, there exists a pair of invariant components of a singular fibre such that they intersect at a point, say p. As this point is at the intersection of two invariant curves, it is fixed by the G-action. Therefore, all the vector fields V_1, V_2, V_3 vanish at the fixed point p. It follows that the section w vanishes at p to order at least three. By the reality of the section w, w also vanishes at the conjugate point $\bar{p}$ in the twistor space to order at least three. Yet the restriction of $\mathbf{K}^{-1}$ onto the real twistor line L through p and $\bar{p}$ has order 4 [9]. Therefore, when the section w vanishes on L to order 6, it vanishes identically along L. However, when the scalar curvature vanishes, the divisors of zeroes of the anti-canonical bundle $\mathbf{K}^{-1}$ are transversal to all real twistor lines becase they are horizontal [33]. This contradiction shows that M has to be minimal.

As observed in [4], a minimal scalar-flat Kähler surface is isometrically covered by one of the following: a flat torus, a $K3$-surface with the Calabi-Yau metric, or a conformally flat Kähler metric on a ruled surface of genus $g \geq 2$.

The $K3$-surfaces are excluded from our investigation because they do not admit any holomorphic vector fields. On the other hand, if (M, g) is a conformally flat Kähler metric on a ruled surface, its Riemannian universal covered is $S^2 \times \mathscr{H}$ with the metric of constant curvature -1 on the upper half plane $\mathscr{H}$ and the round sphere metric of constant curvature $+1$ on S^2. Therefore, (M, g) is finitely covered by $S^2 \times \Sigma_g$ equipped with the product metric [19].

(iii) Finally, if (X, g) is finitely covered by the flat torus and admits a 3-dimensional group of automorphisms, due to Pak's classification [30], (X, g) is the flat torus. (see Theorem (5.1) below).

5. Symmetry of Positive Type Self-dual Metrics

Since the symmetry group G is assumed to be compact, we can assume that it is a group of isometries with respect to a self-dual metric g on X [14], [26].

In this section, we also assume that the self-dual conformal class on X is positive type. This assumption immediately imposes a topological constraint on the manifold X, namely, the intersection form being positive-definite [15]. On the other hand, a topological classification of four-dimensional oriented manifolds with large compact group of diffeomorphisms is also available.

Theorem 5.1. [30] *When X is a compact oriented four-dimensional manifold supporting an effective T^3-action, then X is one of the following*:

(1) *the* 4-*torus* $S^1 \times S^1 \times S^1 \times S^1$,
(2) $S^2 \times S^1 \times S^1$,
(3) $S^3 \times S^1$,
(4) $L(p, q) \times S^1$, *where* $L(p, q)$ *is a lens space.*

Theorem 5.2. [22] *When X is a compact oriented four-dimensional manifold supporting an effective action of a compact non-abelian Lie group, then X is one of the following*:

(1) S^4, *the sphere*;
(2) $\mathbf{CP}^2$, *the complex projective plane*;

(3) *equivariantly finitely covered by* $S^1 \times SU(2)$ *or* $S^1 \times SO(3)$;
(4) *connected-sums of copies of* $S^1 \times S^3$ *or* $S^1 \times \mathbf{RP}^3$;
(5) S^2*-bundle over surfaces*;
(6) *certain quotients of* S^2*-bundles over surfaces by involution.*

(i) The condition on the intersection form rejects items (1) and (2) of Theorem 5.1 as a candidate of positive type self-dual manifolds. In the case of 3-torus symmetry, the only candidates are the Hopf manifold $S^1 \times S^3$ and its finite quotients $S^1 \times L(p, q)$. According to the proof in [30], the group action of S^1 on the S^1-factor in the product manifolds $S^1 \times S^3$ and $S^1 \times L(p, q)$ is free. By our choice of metric, this 1-dimensional group also acts isometrically. Therefore, the induced metric on each orbit of this S^1-action is an Euclidean circle. Taking the projection from the 4-manifold onto the orbit space of this action, we have

$$\pi : S^1 \times N^3 \to N^3$$

where N^3 is either S^3 or the lens space $L(p, q)$. As π is to take the quotient of the manifold by a group of isometries, there is a naturally defined metric g_{N^3} on N^3 such that the projection π is a Riemannian submersion. Then the metric on $S^1 \times N^3$ has the form

$$r g_{S^1} + \pi^* g_{N^3},$$

where g_{S^1} is the Euclidean circle metric on S^1 with radius 1 and r is a function on N^3. Obviously, this metric is conformal to

$$g_{S^1} + \pi^* \left(\frac{1}{r} g_{N^3} \right).$$

Therefore, the conformal structure on $S^1 \times N^3$ is a Riemannian product of the radius 1 circle and a metric on N^3.

Moreover, since the signature of the manifold $S^1 \times N^3$ is equal to zero, a conformal class on this manifold is self-dual if and only if it is conformally flat [3]. In order to have a conformally flat metric on a Riemannian product such as the Euclidean circle with N^3, it is necessarily that the metric $\frac{1}{r} g_{N^3}$ on N^3 has constant sectional curvature [14]. In particular, it is finitely covered by the product of the Euclidean sphere metric on $S^1 \times S^3$.

(ii) When the symmetry group is non-abelian, the candidates of manifolds admitting self-dual conformal classes are described by Theorem 5.2. When the intersection form is required to be positive-definite, we can reject items (5) and (6) from the list. The reason is the following: these two items arised in Theorem 3.7 of [22]. Using the notations and the definitions of [22], the manifolds in question are

$$M(S^2), \qquad N(S^2), \qquad M(P^2), \qquad N(P^2),$$

$$M(F) \quad \text{if } \partial F = \phi, \qquad P(F) \quad \text{if } \partial F \neq \phi.$$

When the compact surface F has non-empty boundary, one may take its double along its boundary:

$$DF := F \bigcup_{\partial} (-F).$$

Then $P(F)$ is doubly covered by $M(DF)$. Since any self-dual conformal class on $P(F)$ can be lifted to be a self-dual conformal class on $M(DF)$, we consider manifolds of the type $M(F)$ with $\partial F = \phi$. In this case, if F is not orientable, we may consider its orientable double covering $F^{\sim}$. Then $M(F)$ is doubly covered by $M(F^{\sim})$. Therefore, we can assume that F is orientable. When F is an orientable compact surface without boundary, $M(F)$ is diffeomorphic to $S^2 \times F$ [22]. However, such manifold cannot admit positive type self-dual conformal class because its intersection form is indefinite.

Similarly, $M(P^2)$ and $N(P^2)$ are doubly covered by $M(S^2)$ and $N(S^2)$ respectively. By definition [22], $M(S^2)$ is diffeomorphic to $S^2 \times S^2$ and $N(S^2)$ is diffeomorphic to the blow-up of the complex projective plane at one point: $\mathbf{CP}^2 \# \overline{\mathbf{CP}}^2$. Since the intersection forms of these manifolds are indefinite, there is not any positive type self-dual conformal class on such manifolds.

(iii) The above observation means that the proof of Theorem (1.3) is completed if we can identify the self-dual conformal classes on the manifolds described in items (1), (2), (3) and (4) in Theorem 5.2.

(iv) As the signature of the four-sphere is equal to zero, any self-dual conformal class on the four-sphere is conformally flat. Due to Kuiper's theorem [11], a conformally flat conformal class on S^4 must contain the Euclidean sphere metric.

By [Theorem A, 35], a positive type self-dual conformal class on $\mathbf{CP}^2$ must contain the Fubini-Study metric.

(v) The manifold $r(S^1 \times S^3) \# p(S^1 \times \mathbf{RP}^3)$ arises in Theorem 5.2 only when there is an SO(3)-action with S^2 as its principal orbits [22]. The singular orbit types are the real projective space $\mathbf{RP}^2$ and the fixed points. The orbit space $\bar{\Sigma}$ is a compact connected 2-dimensional oriented surface such that the boundary components of $\bar{\Sigma}$ are copies of S^1 parametrizing singular orbits. When the singular orbits on X are removed, the 4-manifold is diffeomorphic to $S^2 \times \Sigma$, where Σ is the interior of $\bar{\Sigma}$. Since the group SO(3) acts as a group of isometries, the induced metric on each principal orbit is the Euclidean sphere metric. However, the sectional curvature of the principal orbits is a function of the orbit space Σ. When the metric on $S^2 \times \Sigma$ is conformally changed by a division by the volume of the principal orbits, the induced metric on each copy of S^2 has sectional curvature 1.

On the other hand, the projection

$$S^2 \times \Sigma \to \Sigma$$

is a Riemannian submersion defined by the orbits of the group of isometries. When the induced metric on each fibre of this submersion is independent of Σ, the metric on $S^2 \times \Sigma$ is a Riemannian product.

Moreover, the product metric on $S^2 \times \Sigma$ is a conformal change of the restriction of a self-dual metric on $r(S^1 \times S^3) \# p(S^1 \times \mathbf{RP}^3)$. Since the signature of the manifold $r(S^1 \times S^3) \# p(S^1 \times \mathbf{RP}^3)$ is equal to zero, any self-dual metric on this manifold is conformally flat. Therefore, the product metric on $S^2 \times \Sigma$ is also conformally flat. Since the sectional curvature on S^2 is equal to 1, it is necessarily that the sectional curvature on Σ is equal to -1 [14]. Therefore, the self-dual conformal class is the $C(r, p)$ defined in (1.4).

(vi) Finally, we shall determine the conformal structure on $S^1 \times \mathrm{SU}(2)$ and $S^1 \times \mathrm{SO}(3)$. In terms of equivariant classification, these two manifolds are different from the $C(1,0)$ and $C(0,1)$ because $S^1 \times \mathrm{SU}(2)$ and $S^1 \times \mathrm{SO}(3)$ arise in item (3) of Theorem 5.2 as codimension 1 manifolds, i.e. the principal orbits have codimension 1. In fact, the SU(2) and SO(3) action on $S^1 \times \mathrm{SU}(2)$ and $S^1 \times \mathrm{SO}(3)$ respectively are free. Using the usual double covering of SO(3) by SU(2), we may consider $S^1 \times \mathrm{SU}(2)$ as an SU(2)-equivariant double covering of $S^1 \times \mathrm{SO}(3)$. Although the SU(2)-action on SO(3) is not effective, we do not care as far as the conformal geometry on $S^1 \times \mathrm{SO}(3) \cong S^1 \times \mathbf{RP}^3$ is concerned.

The universal covering of $S^1 \times \mathrm{SU}(2)$ is $\mathbf{R} \times S^3$. The conformally flat structure on $S^1 \times \mathrm{SU}(2)$ as well as the SU(2) group of conformal transformations are lifted onto $\mathbf{R} \times S^3$. Since the conformal class on $S^1 \times \mathrm{SU}(2)$ is positive type, the developing map conformally embeds $\mathbf{R} \times \mathrm{SU}(2)$ into the Euclidean 4-sphere S^4 [39]. By Liouville's theorem, the group of conformal transformations SU(2) on $\mathbf{R} \times \mathrm{SU}(2)$ is extended to be a compact subgroup of the group of conformal transformations of the Euclidean 4-sphere [14], [34]. When a principal orbit of the SU(2)-action on S^4 is the 3-dimensional sphere, there are two fixed points [22], [23]. When S^4 is considered as the projective quaternionic space $\mathbf{HP}^1$, after possibly a Möbius transformation on S^4, we can assume that these two fixed points are $[1,0]$ and $[0,1]$ where $[q_0, q_1]$ is the homogeneous quaternionic coordinates. As the group SU(2) has to leave these two points fixed, it is one of the following subgroup of $PGL(2, \mathbf{H})$:

$$\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix},$$

where $q \in Sp(1) \cong \mathrm{SU}(2)$. It follows that the SU(2)-orbits are the Euclidean 3-spheres.

Both cases are similar to the situation discussed in Example 2.2. Due to Example 2.2, the holomorphic distribution $\mathscr{D}$ defined by the given SU(2)-action on the twistor space of S^4 is rank 2. This distribution descends to be a rank 2 distribution sheaf on the twistor space of $S^1 \times \mathrm{SU}(2)$. The divisors of the holomorphic sections s_1, s_2, s_3 defined in (3.2) are discrete quotients of (some open subsets of) the zero divisors of the homogeneous polynomials defined in (2.4). In particular, if we use the twistor fibration to pull-back the metric on $S^1 \times \mathrm{SU}(2)$ onto one of the two components of the divisor of s_1, we obtain a conformally flat Hermitian structure on $S^1 \times \mathrm{SU}(2)$.

Conformally flat Hermitian surfaces are classified by Pontecorvo [34]. In particular, it is proved that such metric on $S^1 \times SU(2)$ must be the product of the Euclidean sphere metric on S^1 and on S^3. The proof is a careful analysis of the twistor space over $S^1 \times S^3$ and an analysis of the effective divisors given by the zeroes of the section s_1. It was done in Sec. 3 of [34]. We do not repeat the argument here.

Remark 5.3. To prove Theorem 1.2 and Theorem 1.3, one can completely avoid the application of the topological classifications given in Theorem 5.1 and Theorem 5.2. The reason is that, due to the proof of Theorem 1.1, there are effective divisors in the twistor spaces when the dimension of the group of conformal transformations is at least 3. One may analyse the structure of these divisors as in the proof of Theorem 1.2 and in the work of [34] and [35] to come up with a geometrical classification along with the necessary topological classification. This method will rely on the Enriques-Kodaira classification of compact complex surfaces.

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