

ADJUNCTION MAPPINGS ON NUMERICAL QUADRICS

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Abstract.

Using Reider's method, we analyse the adjunction mappings on general-type surfaces with $p_g = 0$, $q = 0$, $c_2 = 4$ and even intersection form. In particular, we prove that when the fundamental group of such a surface does not have any irreducible $SU(2)$ or $SO(3)$ -representations, there exists a divisor numerically equivalent to the canonical divisor so that its adjunction map is a regular bimeromorphism.

§1. Introduction.

When F is a line bundle on a compact complex surface X , the adjunction map is the rational map associated to the line bundle KF where K is the canonical line bundle on X [16]. Applying a construction of Serre [5] [8] [15] and Bogomolov's criterion of determining stability of holomorphic vector bundles on compact complex surfaces [3] [13], Reider developed his famous method of studying adjunction maps on algebraic surfaces [14]. In particular, he re-proved and improved Bombieri's results on pluricanonical maps on general-type surfaces [4].

Recently, Kotschick applied Reider's method to study pluricanonical maps of numerical Godeaux surfaces and numerical Campedelli surfaces [9] [10]. Among other results, he found that the space of irreducible $SO(3)$ and $SU(2)$ -representations of the fundamental group of such a surface often contains obstructions to a certain pluricanonical map from being an embedding. The representations of the fundamental group appear because of Donaldson's Stable Bundle Theorem [7] stating that a holomorphic vector bundle on an algebraic surface is stable if and only if there is a Hermitian-Einstein metric.

We use Kotschick's idea of applying Reider's method and Donaldson's Stable Bundle Theorem to study adjunction maps on numerical quadric surfaces. A numerical quadric surface is a general-type surface having the same set of numerical topological data as the complex quadric surface. Let

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b_n be the n th Betti number and χ the Euler number of a compact complex surface X . Let Q be the intersection form on the torsion-free part of the second integral cohomology group. When the intersection form is odd, we set $\epsilon = 1$. When the intersection form is even, we set $\epsilon = 0$. Let b_+ be the number of positive eigenvalues of the intersection form.

DEFINITION. A general-type surface X is a *numerical quadric surface* if

$$b_+(X) = 1, \quad b_1(X) = 0, \quad \chi(X) = 4, \quad \epsilon(X) = 0.$$

Let p_g be the geometric genus of a compact complex surface X , q the irregularity, c_1^2 the self-intersection of the first Chern class and c_2 the second Chern number. The Hodge theory yields the following identities.

$$b_+ = 2p_g + 1, \quad b_1 = 2q, \quad \chi = c_2.$$

Therefore, we rewrite the definition of numerical quadrics.

DEFINITION. A general-type surface X is a *numerical quadric surface* if

$$p_g(X) = 0, \quad q(X) = 0, \quad c_2(X) = 4, \quad \epsilon(X) = 0.$$

By Qin's answer to the Hirzebruch problem in the category of differentiable manifolds [12], no numerical quadric is diffeomorphic to a quadric surface. Whether there are fake quadrics, i.e. general-type surfaces homeomorphic to but not diffeomorphic to a quadric, remains a question. However, when one allows the fundamental group to be non-trivial, there are examples of numerical quadrics constructed by Kuga and Beauville [1] [2]. These examples are discussed in detail by Dolgachev in [6]. Note that in [1], numerical quadrics were called fake quadrics.

It is a direct consequence of Reider and Bombieri's result that the third canonical map on a numerical quadric is an embedding and that the second canonical map is a regular morphism. The aim of this article is to prove the following two theorems.

THEOREM A. *Let X be a numerical quadric. If $\pi_1(X)$ does not have any irreducible $SU(2)$ or $SO(3)$ -representations, then there exists a divisor F numerically equivalent to the canonical divisor such that the adjunction map of F is a regular bimeromorphism.*

THEOREM B. *Let F be a divisor on a numerical quadric. Suppose that it is not numerically equivalent to the canonical class.*

(1) *If $F - K$ is numerically effective, then the adjunction map of F is a regular bimeromorphism.*

(2) *If $F - K$ is positive, then the adjunction map of F is an embedding.*

Other related results are discussed in the last section of this article.

§2. Preliminary data.

To set up our investigation, we need to describe effective divisors and positive divisors on a numerical quadric. It was already done in Qin's paper, at least when the manifold X is simply-connected [12]. In this section, we follow his techniques to describe divisors which are positive, effective and numerically effective.

When X is a numerical quadric, its second betti number is equal to 2 and its signature is equal to zero. Since the intersection form of X is even, there exists a basis in the torsion-free part of $H^2(X, \mathbb{Z})$ such that the intersection form is the hyperbolic matrix

$$(2.1) \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

LEMMA 2.2. *A numerical quadric does not contain any smooth rational curves. It is a minimal surface and its canonical bundle is ample.*

PROOF. Since the intersection form of a numerical quadric X is even, X is minimal. On a minimal general-type surface, the number of disjoint non-singular rational curves is bounded by $\frac{2}{3}(c_2 - \frac{1}{3}c_1^2)$ [11]. Since $c_2 = 4$ and the signature is equal to zero, the signature formula shows that $c_1^2 = 8$. Then the number of disjoint rational curves on X is at most $\frac{8}{9}$. In other words, there are no rational curves on X .

The canonical bundle is positive because the only obstruction to the canonical bundle being positive on any general-type surface is the presence of smooth rational curves with self-intersection -2 .

On X , two divisors A and B are numerically equivalent if for any divisor D , $A \cdot D = B \cdot D$. By $A \equiv B$, we mean that A and B are numerically equivalent. Consider the induced long exact sequence of the exponential sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 1.$$

The vanishing of the geometric genus and irregularity together implies that

$$H^1(X, \mathcal{O}^*) \cong H^2(X, \mathbb{Z}) \cong H^{1,1}(X).$$

The notions of holomorphic line bundles, first Chern class of line bundles and divisor classes are equivalent. For this reason, we shall not use different symbols to distinguish them. Warning: Given a line bundle F or its divisor, we shall use F^2 to represent both the tensor product of line bundles $F \otimes F$

and the self-intersection number of a divisor $F \cdot F$. In contents, it will not be confusing.

Let Σ and Θ be elements in $H^2(X, \mathbb{Z})$ such that they span the torsion-free part of $H^2(X, \mathbb{Z})$ and the intersection matrix with respect to this basis is represented by (2.1). Then there are integers a and b such that $K \equiv a\Sigma + b\Theta$. As $K^2 = c_1^2 = 8$, $ab = 4$. Moreover, for any divisor D ,

$$\begin{aligned} K \cdot D &= w_2(X) \cdot D, \text{ mod } 2 \\ &= D \cdot D, \text{ mod } 2 \\ &= 0, \text{ mod } 2 \end{aligned}$$

The integers a and b are both even. It follows that either $a = b = 2$ or $a = b = -2$. In the later case, if we choose $\{-\Sigma, -\Theta\}$ as a basis for the torsion-free part of $H^2(X, \mathbb{Z})$, then $K \equiv 2(-\Theta) + 2(-\Sigma)$. With respect to this basis, the intersection form is again represented by the hyperbolic matrix (2.1). To conclude, we have

LEMMA 2.3. *There is a basis $\{\Theta, \Sigma\}$ for the torsion-free part of $H^2(X, \mathbb{Z})$ such that the intersection form is the hyperbolic matrix (2.1) and $K \equiv 2\Theta + 2\Sigma$.*

We choose such a basis for the torsion-free part of $H^2(X, \mathbb{Z})$ once and for all.

LEMMA 2.4. [12] *If a divisor D is effective, then there are non-negative integers a and b such that $D \equiv a\Theta + b\Sigma$ and $a + b > 0$.*

PROOF. It suffices to prove this claim for irreducible effective divisors. As the canonical bundle is positive, $2(a + b) = DK > 0$. Applying the adjunction formula, one sees that if $ab < 0$, then D is a smooth rational curve, contradicting Lemma 2.2.

LEMMA 2.5. *A divisor on X is positive if and only if it is numerically equivalent to $m\Theta + n\Sigma$ such that $m \geq 1$ and $n \geq 1$.*

PROOF. This is a direct consequence of the Nakai's criterion and the above lemma on effective divisors.

COROLLARY 2.6. *A divisor on X is numerically effective if and only if it is numerically equivalent to $m\Theta + n\Sigma$ such that $m \geq 0$ and $n \geq 0$.*

PROOF. For any positive integers h and k , $\Theta + h\Sigma$ and $k\Theta + \Sigma$ are ample. Therefore, if $C \equiv m\Theta + n\Sigma$ is numerically effective, then

$$mh + n = C(\Theta + h\Sigma) \geq 0, \quad m + nk = C(k\Theta + \Sigma) \geq 0,$$

for all positive integers h and k . It follows that m and n are non-negative.

§3. Linear systems.

We apply the Riemann-Roch formula to find that when $D \equiv m\Theta + n\Sigma$,

$$(3.1) \quad \chi(X, D) = \frac{1}{2}(m\Theta + n\Sigma)((m-2)\Theta + (n-2)\Sigma) + 1 = (m-1)(n-1).$$

By Serre duality $h^2(X, D) = h^0(X, -(D - K))$. Therefore, if $D \neq K$ and $D - K$ is numerically effective, then Lemma 2.4 and Lemma 2.5 together implies that $h^2(X, D)$ is equal to zero. Then (3.1) implies that $h^0(X, D) \geq 2$. When F is a line bundle, we use Φ_{KF} to denote the adjunction map. i.e. the rational map associated to the complete linear system of $|K + F|$.

When F is numerically effective and non-zero, the above discussion shows that the system $|K + F|$ is at least a pencil. The following is a direct consequence of Part (i) of Reider's Theorem 1 [14].

PROPOSITION 3.2. *Suppose that F is a numerically effective divisor on X such that $F \equiv (m\Theta + n\Sigma)$ and $mn \geq 3$. If the system $|K + F|$ has a base point then $m = 1$ or $n = 1$. And if p is a base point, then it is contained in an effective divisor E such that $E \equiv \Theta$ when $F \equiv m\Theta + \Sigma$, and $E \equiv \Sigma$ when $F \equiv \Theta + n\Sigma$.*

To illustrate the involvement of the representations of the fundamental group, we prove the following.

PROPOSITION 3.3. *Suppose that the fundamental group of X does not have any irreducible $SO(3)$ -representations. Let F be a numerically effective divisor on X such that $F \equiv (m\Theta + n\Sigma)$ and $mn = 2$. If p is a base point, then it is contained in an effective divisor E such that $E \equiv \Theta$ when $F \equiv 2\Theta + \Sigma$, and $E \equiv \Sigma$ when $F \equiv \Theta + 2\Sigma$.*

PROOF. Let p be a base point of the system $|K + F|$. Let \mathcal{I}_Z be the ideal sheaf of p . By a construction of Serre [5], there is a rank-2 holomorphic bundle \mathcal{E} defined by the extension

$$(3.4) \quad 0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z(F) \rightarrow 0$$

As $c_1(\mathcal{E}) = F$ and $c_2(\mathcal{E}) = 1$, the discriminant of \mathcal{E} is given by

$$c_1^2(\mathcal{E}) - 4c_2(\mathcal{E}) = 2mn - 4 = 2(mn - 2) = 0.$$

When $mn = 2$, $w_2(F) \neq 0$. Due to a generalization of Donaldson's Stable Bundle Theorem [7] [10], the stability of the bundle \mathcal{E} is due to the existence of an anti-self-dual connection on the $SO(3)$ -bundle $\wp := \text{Ad}(\mathcal{E})$ with Pontryagin class

$$p_1(\wp) = c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) = 0.$$

As the connection is also anti-self-dual, Chern-Weil theory implies that the connection on \wp is flat. Therefore, the holonomy representation of \wp produces an irreducible $SO(3)$ -representation of $\pi_1(X)$.

With the absence of such a representation, the bundle \mathcal{E} is unstable with respect to any polarization. Let M be a maximally destabilizing sheaf with respect to the bundle F . Then there is a 0-cycle A and a line bundle E such that \mathcal{E} is contained in the following extension.

$$(3.5) \quad 0 \rightarrow \mathcal{O}(M) \rightarrow \mathcal{E} \rightarrow \mathcal{I}_A(E) \rightarrow 0.$$

As M is destabilizing with respect to F ,

$$(3.6) \quad MF \geq \frac{1}{2}c_1(\mathcal{E})F = \frac{1}{2}F^2 > 0.$$

Therefore, $H^0(X, \mathcal{O}(M^{-1}))$ vanishes. When (3.5) is twisted by M^{-1} to

$$(3.7) \quad 0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \otimes M^{-1} \rightarrow \mathcal{I}_A(EM^{-1}) \rightarrow 0,$$

we see that the bundle $\mathcal{E} \otimes M^{-1}$ has non-trivial sections. When (3.4) is twisted, one has the exact sequence

$$(3.8) \quad 0 \rightarrow \mathcal{O}(M^{-1}) \rightarrow \mathcal{E} \otimes M^{-1} \rightarrow \mathcal{I}_Z(FM^{-1}) \rightarrow 0.$$

Its induced long exact sequence and the vanishing of $H^0(X, \mathcal{O}(M^{-1}))$ together shows that p is contained in an effective divisor E such that $E = F - M$. Note that (3.6) is equivalent to $(M - E)F = (2M - F)F \geq 0$. We have $M^2 \geq E^2$. By Lemma 2.4, $M^2 \geq E^2 \geq 0$.

If $M^2 = E^2 = 0$, then $ME > 0$ because $F^2 = (M + E)^2 > 0$. If $M^2 > 0$, then by Lemma 2.5, either M is positive or $-M$ is positive. By (3.6), M is positive. As E is effective, then $ME > 0$. In conclusion, $ME > 0$ for all cases.

From (3.4) and (3.5), the second Chern class of \mathcal{E} is equal to $1 = c_2(\mathcal{E}) = ME + \deg A$. Therefore, $ME = 1$ and $\deg A = 0$. Given $ME = 1$ and $F = M + E$, with Lemma 2.4 and Corollary 2.6, one deduces that either $E \equiv \Theta$ and $M \equiv (m - 1)\Theta + \Sigma$, or $E \equiv \Sigma$ and $M \equiv \Theta + (n - 1)\Sigma$.

The following is a direct consequence of part (ii) of Reider's Theorem 1 [14]. We give some details to the proof as it helps to understand other related observations.

PROPOSITION 3.9. *Suppose that F is a numerically effective divisor on X such that $F \equiv (m\Theta + n\Sigma)$ and $mn \geq 5$. If Z is a 0-cycle of degree two not separated by sections of the bundle KF , then $m = 1$ or $n = 1$. And the support of Z is contained in an effective divisor E such that*

- (1) $E \equiv \Theta$ or 2Θ when $F \equiv m\Theta + \Sigma$, $E \equiv \Sigma$ or 2Σ when $F \equiv \Theta + n\Sigma$; or
- (2) $E \equiv \Theta$ when $F \equiv m\Theta + 2\Sigma$, $E \equiv \Sigma$ when $F \equiv 2\Theta + n\Sigma$.

♦

PROOF. Given Z , there is a rank-2 holomorphic bundle \mathcal{E} defined by the extension

$$(3.10) \quad 0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z(F) \rightarrow 0$$

As $c_1(\mathcal{E}) = F$ and $c_2(\mathcal{E}) = \deg Z = 2$, the discriminant of \mathcal{E} is equal to

$$(3.11) \quad c_1^2(\mathcal{E}) - 4c_2(\mathcal{E}) = 2mn - 8 = 2(mn - 4).$$

With the assumption that $mn \geq 5$, the bundle \mathcal{E} is Bogomolov-unstable [3] [13]. Let M be a maximally destabilizing sheaf with respect to the bundle F . Then there is a 0-cycle A and a line bundle E such that \mathcal{E} is contained in the following extension.

$$0 \rightarrow \mathcal{O}(M) \rightarrow \mathcal{E} \rightarrow \mathcal{I}_A(E) \rightarrow 0.$$

As M is destabilizing with respect to F , $MF \geq \frac{1}{2}c_1(\mathcal{E})F = \frac{1}{2}F^2 > 0$. In particular, $H^0(X, \mathcal{O}(M^{-1}))$ vanishes. Applying the argument in the second half of the proof of Proposition 3.3, one finds an effective divisor E such that $E = F - M$ and support of Z is contained in E and $ME > 0$.

The second Chern class of \mathcal{E} is equal to $2 = c_2(\mathcal{E}) = ME + \deg A$. Since $ME > 0$, either $ME = 1$ or $ME = 2$. With Lemma 2.4 and Corollary 2.6, one deduces the following. When $ME = 1$, then $E \equiv \Theta$, $M \equiv (m-1)\Theta + \Sigma$; or $E \equiv \Sigma$, $M \equiv \Theta + (n-1)\Sigma$. When $ME = 2$, then $E \equiv \Theta$, $M \equiv (m-1)\Theta + 2\Sigma$; or $E \equiv \Sigma$, $M \equiv 2\Theta + (n-1)\Sigma$; or $E \equiv 2\Theta$, $M \equiv (m-2)\Theta + \Sigma$; or $E \equiv 2\Sigma$, $M \equiv \Theta + (n-2)\Sigma$.

It is convenient to introduce the following definitions.

DEFINITION 3.12. *Let F be a numerically effective divisor. Let Z be any 0-cycle with finite degree. When a rank-2 vector bundle is obtained in the extension*

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z(F) \rightarrow 0,$$

it is called an extension bundle of the divisor F with respect to Z .

Suppose that \mathcal{E} is an extension bundle. If \mathcal{E} is unstable with respect to a polarization, then there is a maximally destabilizing line bundle M and a 0-cycle A of finite degree such that \mathcal{E} is contained in the following destabilizing exact sequence.

$$0 \rightarrow \mathcal{O}(M) \rightarrow \mathcal{E} \rightarrow \mathcal{I}_A(FM^{-1}) \rightarrow 0.$$

DEFINITION 3.13. *A divisor E such that $E = F - M$ is called a degeneracy divisor.*

Reading the proof of Proposition 3.9 again. Equality (3.11) shows that one cannot apply Bogomolov's theorem if $mn = 4$. Suppose that the extension bundle \mathcal{E} is stable. If F has a square root $F^{1/2}$ in the Picard group, then $\mathcal{F} = \mathcal{E} \otimes F^{-1/2}$ is a stable bundle with vanishing Chern classes. Then Donaldson's Hermitian-Einstein metric on \mathcal{F} is a flat $SU(2)$ -metric. If F does not have square root, then $\text{Ad}(\mathcal{E})$ is a flat $SO(3)$ -bundle. Therefore, the holonomy representations of such bundles determine irreducible $SU(2)$ and $SO(3)$ -representations of $\pi_1(X)$ respectively. Then from the proof of Proposition 3.3, we have the following.

PROPOSITION 3.14. *Suppose that the fundamental group of X does not have any irreducible $SO(3)$ -representations. Suppose that F is a divisor such that $F \equiv (4\Theta + \Sigma)$ or $(\Theta + 4\Sigma)$. If Z is a 0-cycle of degree two not separated by sections of the bundle KF , then Z is contained in an effective divisor E such that $E \equiv \Theta$ or $E \equiv 2\Theta$ when $F \equiv 4\Theta + \Sigma$; and $E \equiv \Sigma$ or $E \equiv 2\Sigma$ when $F \equiv \Theta + 4\Sigma$.*

PROPOSITION 3.15. *Suppose that the fundamental group of X does not have any irreducible $SO(3)$ or $SU(2)$ -representations. Suppose that F is a divisor such that $F \equiv K$. If Z is a 0-cycle of degree two not separated by sections of the bundle KF , then Z is contained in an effective divisor E such that $E \equiv \Theta$, Σ , or $\Theta + \Sigma$.*

§4. Proof of The Main Theorems.

All propositions in the last section focus our attention on *effective* divisors E numerically equivalent to Σ , Θ or $\Theta + \Sigma$. In order to prove Theorem A and Theorem B in the introduction, we investigate the properties of such divisors.

LEMMA 4.1. *Suppose that E is a divisor numerically equivalent to Θ or Σ . Then $\dim|E| \leq 0$.*

PROOF. We prove this claim when $E \equiv \Sigma$. Suppose that $\dim|E| \geq 1$. Due to Lemma 2.4, every element in this system $|E|$ is irreducible. As $EE = \Sigma\Sigma = 0$, any two elements in this system are mutually disjoint. In particular, this system is free of base points. By Bertini's theorem, a generic element in this system is a non singular irreducible curve. Let D be such an element. Then one has the exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(E) \rightarrow \mathcal{O}_D(E) \rightarrow 0.$$

Since $EE = 0$ and $\dim|E| \geq 1$, $\mathcal{O}_D(E) = \mathcal{O}_D$. Therefore, $q = 0$ implies that $|E|$ is a pencil.

Let D be any nonsingular irreducible element in the pencil $|E|$. By the ad-

junction formula, the canonical divisor K_D of the curve D is the restriction of $K + E$ and the genus of D is equal to 2. It follows that

$$h^0(D, \mathcal{O}_D(K_D)) = 2, \quad h^1(D, \mathcal{O}_D(K_D)) = h^0(D, \mathcal{O}_D) = 1.$$

As $p_g = 0$ and $q = 0$, we deduce an isomorphism

$$H^0(X, \mathcal{O}(KE)) \cong H^0(D, \mathcal{O}_D(K_D)) \cong \mathbb{C}^2$$

from the following exact sequence.

$$0 \rightarrow \mathcal{O}(K) \rightarrow \mathcal{O}(KE) \rightarrow \mathcal{O}_D(K_D) \rightarrow 0.$$

It means that the system $|K + E|$ is a pencil. As $(K + E)^2 = 12$, this system has base points. Let p be a base point. Let G be the unique element in the pencil $|E|$ through p . If G is singular at p , let q be any point of G different from p . As the system $|K + E|$ is a pencil, there is an element C in this system passing through q . The intersection multiplicity of C and G is at least one at q and two at p . Yet, $(K + E) \cdot G = 2$. Since G is irreducible, G is a component of C . Therefore, $K = C - G$ is effective. This is a contradiction to the vanishing of the geometric genus of X . So G is not singular at p . Similarly, let q be any singular point of G . This point is different from p . Let C be the unique element in the pencil $|K + E|$ through q , then G and C intersect at least once at p and twice at q . Again, we deduce that G is a component of C . Then $K = C - G$ is effective. This contradiction to the vanishing of p_g shows that G is non-singular. When G is non-singular, the conclusion of the last paragraph gives the following isomorphism.

$$H^0(X, \mathcal{O}(KE)) \cong H^0(G, \mathcal{O}_G(K_G)) \cong \mathbb{C}^2.$$

As p is a base point of the system $|K + E|$ and is contained in the curve G , the above isomorphism obtained by restriction implies that p is a base point of the canonical system of the curve G . But G is a non-singular curve with genus 2, its canonical system is base-point-free. This contradiction shows that $|E|$ cannot be a pencil.

LEMMA 4.2. *Suppose that $E_1 \equiv \Theta + \Sigma$. Suppose that the map $\Phi_{KE_1^2}$ is not an embedding. Let E_2 be an effective degeneracy divisor of the system $|K + 2E_1|$. Then $2E_1 \neq 2E_2$. In particular, $E_1 \neq E_2$.*

PROOF. We only need to investigate the case when $E_2 \equiv \Theta + \Sigma$. In this case, the maximally destabilizing exact sequence is

$$0 \rightarrow \mathcal{O}(E_1^2 E_2^{-1}) \rightarrow \mathcal{E} \rightarrow \mathcal{O}(E_2) \rightarrow 0.$$

Therefore, the bundle \mathcal{E} is determined by a non-trivial element in $H^1(X, E_1^2 E_2^{-2})$. As $q = 0$, the bundle $E_1^2 E_2^{-2}$ cannot be trivial.

PROOF OF THEOREM A. Let F be any divisor such that $F \equiv K$. Proposition 3.2 shows that the map Φ_{KF} is regular. Suppose that Φ_{KF} fails to be an embedding. Let Z be a degree-2 0-cycle not separated by sections of the bundle KF . Given the assumption on the fundamental group, the extension bundle of F with respect to Z is unstable. Proposition 3.15 shows that the support of Z is contained in an effective degeneracy divisor.

If all effective degeneracy divisors are numerically equivalent to either Σ or Θ , then Φ_{KF} is a bimeromorphism because the regularity of X implies that there are only finitely many such divisor classes. By Lemma 4.1, there is at most one effective element in the complete linear system of such a divisor class. Therefore, Φ_{KF} fails to be a bimeromorphism only when there is a degeneracy divisor E_1 such that $E_1 \equiv \Theta + \Sigma$ and $\dim|E_1| = 1$.

Given such E_1 , define F_1 by E_1^2 . Consider the adjunction map of F_1 . Suppose that this map is not a bimeromorphism. Applying the conclusion of the last paragraph to F_1 , one finds a degeneracy divisor E_2 such that $E_2 \equiv \Theta + \Sigma$ and $\dim|E_2| = 1$.

Due to Lemma 4.2, $E_1^2 \neq E_2^2$. Define $F_2 := E_2^2$. Φ_{KF_2} must be a bimeromorphism. For otherwise, there is a degeneracy divisor E_3 such that $E_3 \equiv \Theta + \Sigma$ and $\dim|E_3| = 1$. By Lemma 4.2, $E_1 \neq E_2$ and $E_2 \neq E_3$. If $E_1 = E_3$, then

$$H^0(X, E_1 E_2 E_3) = H^0(X, E_1^2 E_2) \supseteq S^2 H^0(X, E_1) \otimes H^0(X, E_2) \equiv C^6.$$

If $E_1 \neq E_3$, then

$$H^0(X, E_1 E_2 E_3) \supseteq H^0(X, E_1) \otimes H^0(X, E_2) \otimes H^0(X, E_3) \equiv C^8.$$

Since $p_g = 0$ and $h^0(X, E_1 E_2 E_3) \geq 6$, $h^2(X, E_1 E_2 E_3) = 0$. The divisors of the bundle $K(E_1 E_2 E_3)^{-1}$ is numerically equivalent to $-(\Theta + \Sigma)$. By Lemma 2.5, it is a negative line bundle. By Kodaira's vanishing theorem and Serre duality, $h^1(X, E_1 E_2 E_3) = 0$. By (3.1), $\chi(X, E_1 E_2 E_3) = 4$. It follows that $h^0(X, E_1 E_2 E_3) = 4$. This is a contradiction to $h^0(X, E_1 E_2 E_3) \geq 6$. Therefore if Φ_{KF_1} is not a bimeromorphism, then Φ_{KF_2} is a bimeromorphism.

The proof of Theorem A is completed.

PROOF OF THEOREM B. When $F - K$ is ample, then $F \equiv m\Theta + n\Sigma$ such that $m \geq 3$ and $n \geq 3$. Therefore, Φ_{KF} is regular because of Proposition 3.2 and is an embedding because of Proposition 3.9.

Suppose that F is not numerically equivalent to K and $F - K$ is numerically effective. If $F - K$ is not ample, then $F \equiv m\Theta + 2\Sigma$ such that $m \geq 3$ or $F \equiv 2\Theta + n\Sigma$ such that $n \geq 3$. Therefore, Φ_{KF} is regular because of Proposition 3.2. Due to part (2) of Proposition 3.9, Lemma 4.1 and the finiteness of

the first integral homology group, one concludes that Φ_{KF} is a bimeromorphism.

The proof of Theorem B is completed.

§5. Other results.

The adjunction maps of F when $F \equiv m\Theta + \Sigma$ and $F \equiv \Theta + n\Sigma$ are yet to be dealt with. We study these maps with an additional topological assumption that the first integral homology group $H_1(X, \mathbb{Z})$ vanishes. Under this condition, two divisors are linearly equivalent if and only if there are numerically equivalent.

PROPOSITION 5.1. *Suppose that X is a numerical quadric such that $H_1(X, \mathbb{Z})$ vanishes and $\pi_1(X)$ does not have any irreducible $SO(3)$ -representations. Let F be $2\Theta + \Sigma$ or $\Theta + 2\Sigma$. Then the system $|K + F|$ has at most one base point.*

PROOF. Assume that $F = 2\Theta + \Sigma$. If p is a base point of the system $|K + F|$. Let \mathcal{E} be the extension bundle of F with respect to p . By Proposition 3.3, Θ is an effective divisor and the maximally destabilizing sequence (3.5) is equivalent to

$$(5.2) \quad 0 \rightarrow \Theta + \Sigma \rightarrow \mathcal{E} \rightarrow \Theta \rightarrow 0.$$

Therefore, \mathcal{E} is defined by a non-trivial element in $H^1(X, \Sigma)$.

As $p_g = 0$, $h^0(X, \Theta + \Sigma) = 0$. Then Lemma 4.1 and exact sequence (5.2) together shows that $h^0(X, \mathcal{E}) \leq 1$. By definition, \mathcal{E} is obtained by the following extension.

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_p(2\Theta + \Sigma) \rightarrow 0.$$

As $q = 0$ and $h^0(X, \mathcal{E}) \leq 1$, the above exact sequence shows that $h^0(X, \mathcal{E}) = 1$ and $h^0(X, \mathcal{I}_p(2\Theta + \Sigma)) = 0$. Therefore, $h^0(X, 2\Theta + \Sigma) = 1$.

Since Θ is effective and the geometric genus of X is equal to zero, $h^0(X, \Sigma) = 0$. Then by (3.1), $h^2(X, \Sigma) = h^1(X, \Sigma)$. By Serre duality, we have $h^0(X, 2\Theta + \Sigma) = h^1(X, \Sigma)$. With the conclusion of the last paragraph, we have $h^1(X, \Sigma) = 1$.

As $h^0(X, \mathcal{E}) = 1$, the zeroes of any non-trivial sections of \mathcal{E} is the point p . As \mathcal{E} is the only non-trivial extension given by $H^1(X, \Sigma)$, when there were any other base points q , its extension bundle would have been the given \mathcal{E} . Moreover, \mathcal{E} would have had a section vanishing at q . Yet $h^0(X, \mathcal{E}) = 1$ and $c_2(\mathcal{E}) = 1$. It shows that $p = q$.

PROPOSITION 5.3. *Suppose that X is a numerical quadric such that $H_1(X, \mathbb{Z})$ vanishes. Let F be $3\Theta + \Sigma$ or $\Theta + 3\Sigma$. Then the adjunction map Φ_{KF} is a regular morphism.*

PROOF. Suppose that $F = 3\Theta + \Sigma$. By Proposition 3.2, when p is a base point of the system $|K + F|$, (3.5) is equivalent to

$$0 \rightarrow \mathcal{O}(2\Theta + \Sigma) \rightarrow \mathcal{E} \rightarrow \mathcal{O}(\Theta) \rightarrow 0.$$

Therefore, \mathcal{E} is determined by a non-trivial element in $H^1(X, \Theta + \Sigma)$.

As the geometric genus of X vanishes, $h^0(X, \Theta + \Sigma) = 0$ and $h^2(X, \Theta + \Sigma) = 0$. Then by the Riemann-Roch formula, $-h^1(X, \Theta + \Sigma) = \chi(X, \Theta + \Sigma) = 0$. It shows that the extension bundle \mathcal{E} does not exist.

PROPOSITION 5.4. *Suppose that X is a numerical quadric such that $H_1(X, \mathbb{Z})$ vanishes and $\pi_1(X)$ does not have any $SU(2)$ -representations. Let F be $4\Theta + \Sigma$ or $\Theta + 4\Sigma$. Then the adjunction map Φ_{KF} is a regular morphism.*

PROOF. Suppose that $F = 4\Theta + \Sigma$. If $|K + F|$ has a base point p , the extension bundle is contained in

$$(5.5) \quad 0 \rightarrow \mathcal{O} \rightarrow \mathcal{E} \rightarrow \mathcal{I}_p(4\Theta + \Sigma) \rightarrow 0.$$

The maximally destabilizing sequence is

$$(5.6) \quad 0 \rightarrow \mathcal{O}(3\Theta + \Sigma) \rightarrow \mathcal{E} \rightarrow \mathcal{O}(\Theta) \rightarrow 0.$$

The last exact sequence shows that \mathcal{E} is determined by a non-trivial class in $H^1(X, 2\Theta + \Sigma)$.

As Θ is effective and $p_g = 0$, $h^0(X, \Sigma) = 0$. Then by Serre duality, $h^2(X, 2\Theta + \Sigma) = 0$. By (3.1), $\chi(X, 2\Theta + \Sigma) = 0$. Therefore, $h^0(X, 2\Theta + \Sigma) = h^1(X, 2\Theta + \Sigma) \neq 0$. Let s be a non-trivial element in $H^0(X, 2\Theta + \Sigma)$. Then s^3 is a section of the bundle KF . In particular, s vanishes at p , a base point of the system $|K + F|$.

Twisting the exact sequences (5.5) and (5.6) by -2Θ , we have

$$(5.7) \quad 0 \rightarrow \mathcal{O}(-2\Theta) \rightarrow \mathcal{E}(-2\Theta) \rightarrow \mathcal{I}_p(2\Theta + \Sigma) \rightarrow 0.$$

and

$$0 \rightarrow \mathcal{O}(\Theta + \Sigma) \rightarrow \mathcal{E}(-2\Theta) \rightarrow \mathcal{O}(-\Theta) \rightarrow 0.$$

The last exact sequence yields the vanishing of $H^0(X, \mathcal{E}(-2\Theta))$. As s is an element in $H^0(X, \mathcal{I}_p(2\Theta + \Sigma))$, then exact sequence (5.7) implies that $H^1(X, -2\Theta) \neq 0$. It follows that there is a non-trivial extension.

$$(5.8) \quad 0 \rightarrow \mathcal{O} \rightarrow \mathcal{F} \rightarrow 2\Theta \rightarrow 0.$$

The discriminant of the bundle \mathcal{F} is equal to zero and the determinant bundle of \mathcal{F} is Θ^2 . When $\pi_1(X)$ does not have any irreducible $SU(2)$ -representations, the bundle \mathcal{F} is unstable with respect to any polarization. Let M be a maximally destabilizing bundle with respect to $\Theta + \Sigma$. One has the

following exact sequence.

$$(5.9) \quad 0 \rightarrow M \rightarrow \mathcal{F} \rightarrow \mathcal{I}_A(2\Theta - M) \rightarrow 0,$$

with

$$(5.10) \quad M(\Theta + \Sigma) \geq \frac{1}{2}c_1(\mathcal{F})(\Theta + \Sigma) = \frac{1}{2}.$$

Therefore, the bundle M^{-1} does not have any non-trivial sections. Due to (5.8) and (5.9), $2\Theta - M$ is effective. Let a and b be non-negative integers such that $2\Theta - M = a\Theta + b\Sigma$. (5.10) implies that $2 - a - b \geq 1$. As $a\Theta + b\Sigma$ is effective, Lemma 2.3 implies that $a + b = 1$. As $c_2(\mathcal{F}) = 0$, $M(2\Theta - M) + \deg A = 0$. Therefore, $2b(1 - a) + \deg A = 0$. Since $a + b = 1$, we have $b = 0$ and $\deg A = 0$. So we re-write (5.9) as

$$0 \rightarrow \Theta \rightarrow \mathcal{F} \rightarrow \Theta \rightarrow 0.$$

Then \mathcal{F} is also defined by a non-trivial element in $H^1(X, \Theta)$. As the surface is regular, such \mathcal{F} cannot exist.

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REFERENCES

1. W. Barth, C. Peters & A. Van de Ven, *Compact Complex Surfaces*, Springer-Verlag, New York, 1984.
2. A. Beauville, *Surfaces Algébriques Complexes*, Astérisque 54, Soc. Math. France, Paris, 1978.
3. F. A. Bogomolov, *Holomorphic tensors and vector bundles on projective varieties*, USSR-Izv. 13 (1978), 499--555.
4. E. Bombieri, *Canonical models of surfaces of general type*, Publ. Math. IHES 42 (1972) 171--219.
5. F. Catanese, *Footnotes to a theorem of I. Reider*, Algebraic Geometry (A. Sommese et. al., eds.), Lecture Notes of Mathematics, 1417 1990.
6. I. Dolgachev, *Algebraic surfaces with $p_g = q = 0$* , Algebraic Surfaces, Liguori, Napoli, 1981.
7. S. K. Donaldson, *Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. 50 (1985) 1--26.
8. P. Griffiths & J. Harris, *Residues and zero-cycles on algebraic varieties*, Ann. of Math. 108 (1978) 461--505.
9. D. Kotschick, *Stable and unstable vector bundles on algebraic surfaces*, Problems in the Theory of Surfaces and Their Classification, INdAM. Academic, New York, 1991.
10. D. Kotschick, *On the pluricanonical maps of Godeaux and Campedelli surfaces*, International J. Math. 5 (1994), 53--60.

11. Y. Miyaoka, *The maximal number of quotient singularities on surfaces with given numerical invariants*, Math. Ann. 268 (1984), 159--171.
12. Z. Qin, *Complex structures on certain differentiable 4-manifolds*, Topology 32 (1993), 551--566.
13. M. Reid, *Bogomolov's theorem $c_1^2 \leq 4c_2$* , International Symp. on Algebraic Geometry, Kyoto, (1977), pp. 623--642.
14. I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math. 127 (1988), 309--316.
15. J.-P. Serre, *Sur les modules projectifs.*, Sèm. Dubreil-Pisot, 2, 1960-1961.
16. A. J. Sommese & A. Van de Ven, *On the adjunction mappings*, Math. Ann. 278 (1987), 593--603.

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