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Superconformal Symmetry and HyperKähler Manifolds with Torsion

Yat Sun Poon[1], **Andrew Swann**[2]

[1] Department of Mathematics, University of California at Riverside, Riverside, CA 92521, USA. E-mail: ypoon@math.ucr.edu
[2] Department of Mathematics and Computer Science, University of Southern Denmark, Campusvej 55, 5230 Odense M, Denmark. E-mail: swann@imada.sdu.dk

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Abstract: The geometry arising from Michelson & Strominger's study of $\mathcal{N} = 4$B supersymmetric quantum mechanics with superconformal $D(2, 1; \alpha)$-symmetry is a hyperKähler manifold with torsion (HKT) together with a special homothety. It is shown that different parameters α are related via changes in potentials for the HKT target spaces. For $\alpha \neq 0, -1$, we describe how each such HKT manifold M^{4m} is derived from a space N^{4m-4} which is quaternionic Kähler with torsion and carries an Abelian instanton.

1. Introduction

In the study of two-dimensional sigma models a variety of different quaternionic geometries arise on the target spaces. In the presence of a Wess-Zumino term the metric connections have non-zero torsion. For $\mathcal{N} = 4$B rigid supersymmetry the target space carries an HKT structure: the geometry of a hyperKähler connection with totally skew symmetric torsion [4]. For $\mathcal{N} = 4$B local symmetry the resulting geometry [6] is known as QKT (quaternionic Kähler with torsion). The mathematical background of HKT geometry was reported in [5], where many examples were constructed. Mathematical discussion of QKT geometry may be found in [7].

Through the work of Maldacena [8] there has been much interest in field theories with superconformal symmetry. Michelson and Strominger [9] showed that for $\mathcal{N} = 4$B rigid supersymmetry examples of quantum mechanical systems in one dimension with actions of the superconformal groups $D(2, 1; \alpha)$ may be obtained. As discussed in [2], $D(2, 1; \alpha)$ has $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ as its algebra of R-symmetries and $D(2, 1; -2)$ is the supergroup $Osp(4|2)$. On the target space, Michelson and Strominger [9] show that the HKT manifold (locally) has a certain vector field X generating one homothety and three isometries, see Eqs. (2.2). In this paper we investigate the geometry of an HKT manifold with such a vector field. In [12], we showed that the length-squared of X gives a potential μ for the HKT metric. By transforming μ we show in Sect. 3 that $D(2, 1; \alpha)$-symmetries for different values of α are closely related. In particular, if an HKT manifold has a vector field X generating a $D(2, 1; \alpha)$-symmetry with $\alpha < 0$ and $\alpha \neq -1$, then the same manifold carries HKT metrics with $D(2, 1; \alpha')$-symmetry for each $\alpha' < 0$. Similarly, any $\alpha' > 0$ may be obtained from any other $\alpha > 0$.

In Sect. 4 we show that the vector fields generate an infinitesimal action of the non-zero quaternions $\mathbb{H}^*$ and that the quotient $N^{4n} = M/\mathbb{H}^*$ carries a QKT metric. It turns out, Sect. 5, that this QKT manifold comes equipped with an instanton connection on its bundle $\Lambda^{4n}T^*N$ of volume forms. Locally QKT metrics inducing instanton connections exist on any quaternionic manifold, and from such a geometry in dimension $4n$ we construct in Sect. 6 HKT metrics with $D(2, 1; \alpha)$-symmetry in dimension $4n + 4$. As an interesting special case, we obtain HKT metrics with $D(2, 1; 1)$-symmetry over each quaternionic Kähler manifold of negative scalar curvature.

Both the discussion of the parameter change for $D(2, 1; \alpha)$-symmetry and the bundle constructions relating QKT and HKT geometries naturally introduce pseudo-Riemannian structures. We therefore deal with HKT geometry in this generality from the outset.

If one sets the torsion to zero in this paper, then one recovers the constructions of [13], relating quaternionic Kähler manifolds to hyperKähler manifolds with $D(2, 1; -2)$-symmetry and hyperKähler potentials. This case is relevant to the discussion of superconformal symmetry in $\mathcal{N} = 2$ quantum mechanics [3].

2. Potentials and Superconformal Symmetry

Let (M, g, I, J, K) be an HKT manifold of dimension $4m$ and signature $(4p, 4q)$. This means that I, J and K are integrable complex structures satisfying the quaternion identities, g is a hyper-Hermitian metric of signature $(4p, 4q)$ and there is an $Sp(p, q)$-connection ∇ whose torsion tensor

$$c(X, Y, Z) = g(X, T(Y, Z))$$

is totally skew, where $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$. The integrability of I implies

$$T(IX, IY) - IT(IX, Y) - IT(X, IY) - T(X, Y) = 0 \tag{2.1}$$

and that c is of type $(2, 1)_I + (1, 2)_I$. Note that for a given (g, I, J, K) there is at most one HKT connection ∇, sometimes called the Bismut connection.

We set $F_I(X, Y) = g(IX, Y)$ and define d_I on r-forms by

$$d_I\beta = (-1)^r IdI\beta,$$

where $I\beta = \beta(-I\cdot, \dots, -I\cdot)$. Similar forms and operators are defined for J and K. With these conventions the torsion satisfies

$$-c = d_I F_I = d_J F_J = d_K F_K.$$

A *potential* for an HKT structure is a function such that

$$F_I = \tfrac{1}{2}(dd_I + d_J d_K)\mu, \quad \text{etc.}$$

Note that $dd_I\mu = dId\mu$ and $d_J d_K\mu = -JdId\mu$.

In [5, Cor. 4] it is shown that locally any hypercomplex manifold (M, I, J, K) admits a compatible HKT metric with potential. On the other hand, Michelson and Strominger

[9, Appendix C] show that for any open set of $\mathbb{R}^{4m} = \mathbb{H}^m$ with the standard complex structures I, J and K, any compatible HKT metric admits a potential. It is an open question whether general HKT structures admit potentials locally. In [12] it was shown that those with $D(2, 1; \alpha)$-symmetry do. Here we summarise that discussion.

Suppose we have an HKT manifold with a vector field X satisfying

$$L_X g = ag, \tag{2.2a}$$
$$L_{IX} g = 0, \quad \text{etc.,} \tag{2.2b}$$
$$L_{IX} I = 0, \quad \text{etc.,} \tag{2.2c}$$
$$L_{IX} J = bK, \quad \text{etc.,} \tag{2.2d}$$

where $a, b \in \mathbb{R}$ are constants and "etc." means that the versions of Eqs. (2.2b–2.2d) obtained by cyclically permuting I, J and K also hold. By rescaling X we may alter the constants a and b, but the point $[a, b] \in \mathbb{R}P(1) \cup \{(0, 0)\}$ remains fixed. We call a vector field X satisfying (2.2) *a special homothety of type* (a, b).

In the notation of [9], HKT geometry with this symmetry arises from a quantum mechanical system with $D(2, 1; \alpha)$-symmetry where $\alpha + 1 = a/b$. For the standard flat metric on $\mathbb{H}^m = \mathbb{R}^{4m}$, the vector field X is given by dilation. This case has $a = 2$, $b = -2$ and $\alpha = -2$.

For a vector field X satisfying Eqs. (2.2a) and (2.2b) on an HKT manifold, it was shown in [12] that

$$\nabla X = \tfrac{1}{2} a \,\mathrm{Id},$$

where ∇ is the torsion connection.

Lemma 2.1. *If $a \neq 0$, then the one-form $X^\flat$ is exact, $g(X, X)$ is non-constant and $X \lrcorner c = 0$.*

Proof. Since ∇ is a metric connection, we have

$$d(g(X, X)) = 2g(\nabla X, X) = ag(\cdot, X) = aX^\flat,$$

using the lemma above. This gives that $X^\flat$ is exact and $g(X, X)$ is non-constant. Now $X^\flat$ is closed, so

$$\begin{aligned} 0 = dX^\flat(Y, Z) &= g(\nabla_Y X, Z) - g(\nabla_Z X, Y) + g(X, T(Y, Z)) \\ &= c(X, Y, Z), \end{aligned}$$

as required. □

Equations (2.2d) and (2.2b) give

$$IX \lrcorner c - J(IX \lrcorner c) = -(a + b)F_I\,, \tag{2.3}$$

which together with the lemma is the basis for the proof of the following result.

Theorem 2.2 ([12])**.** *If X is a special homothety of type (a, b) with $a \neq 0, b$, then the function*

$$\mu = \frac{2}{a(a - b)} g(X, X)$$

is an HKT potential. □

3. Parameter Changes

Suppose M is an HKT manifold with potential μ. If f is a smooth function then Grantcharov and Poon [5] showed that

$$g_f = f'(\mu)\, g + \tfrac{1}{2} f''(\mu)\, (d^{\mathbb{H}}\mu)^2 \tag{3.1}$$

is an HKT metric with potential $f(\mu)$ whenever g_f is non-degenerate, where $(d^{\mathbb{H}}\mu)^2 = d\mu^2 + (I d\mu)^2 + (J d\mu)^2 + (K d\mu)^2$. (Our conventions give slightly different coefficients to those in [5].) Such transformations $\mu \mapsto f(\mu)$ allow us to relate HKT structures with different $D(2,1;\alpha)$-symmetries, perhaps at the cost of changing the signature of the metric.

Proposition 3.1. *Let (M, g, ∇, X) be an HKT manifold with a special homothety X of type (a, b), $a \neq 0, b$, and let μ be the potential found in Theorem 2.2. Suppose f is a smooth real-valued function. Then $f(\mu)$ is a potential for an HKT metric g_f with X a special homothety of type (a', b') if either*

(a) $f(\mu) = |\mu|^k$, $k \neq 0, b/a$, $(a', b') = (ka, b)$ or
(b) $f(\mu) = \log|\mu|$, $(a', b') = (0, b)$.

Up to homothety, these are the only possibilities for g_f.

Note that g_f is only defined away from the set $\mu = 0$.

Proof. It is sufficient to determine when X is a homothety for the metric g_f of (3.1). As $L_X d\mu = a\, d\mu$, we have

$$L_X g_f = a \left(\frac{f''}{f'}\mu + 1 \right) f' g + a \left(\frac{f'''}{f''}\mu + 2 \right) \tfrac{1}{2} f'' (d^{\mathbb{H}}\mu)^2.$$

So X is conformal only if

$$\left(\frac{f''}{f'} - \frac{f'''}{f''} \right) \mu = 1.$$

The left-hand side of this equation is $-\mu \frac{d}{d\mu} \log \left| f''/f' \right|$, so $f''/f' = (k-1)\mu^{-1}$ for some k. As $f''/f' = \frac{d}{d\mu} \log \left| f' \right|$, integration gives $f' = A\, |\mu|^{k-1}$. The constant A only scales g_f by a constant, so we may take $A = \pm 1$. Note that X now scales g_f by a constant. Finally, we may integrate one more time to get the desired functions.

From the form of g_f we have

$$\begin{aligned} g_{|\mu|^k}(X, X) &= \frac{ka - b}{a - b} k \operatorname{sgn}\mu\, |\mu|^{k-1}\, g(X, X), \\ g_{|\mu|^k}(Y, Y) &= k \operatorname{sgn}\mu\, |\mu|^{k-1}\, g(Y, Y), \end{aligned}$$

when Y is g-orthogonal to the quaternionic span of X. Thus $ka - b$ needs to be non-zero to ensure a non-degenerate metric. □

When μ is positive we see that g_{μ^k} has the same signature as g if and only if $ka - b$ has the same sign as $a - b$. For $f(\mu) = \log \mu$ we have

$$g_{\log \mu}(X, X) = \frac{b}{(b-a)\mu} g(X, X), \qquad g_{\log \mu}(Y, Y) = \frac{1}{\mu} g(Y, Y),$$

which has the signature of g only if b and $b - a$ have the same sign. Recalling that $\alpha = -1 + a/b$, we have:

Corollary 3.2. *Let M be a definite HKT manifold with a nowhere zero special vector field generating a $D(2, 1; \alpha)$-symmetry, with α finite.*

(1) If $\alpha > 0$ then for each $\alpha' > 0$ the hypercomplex manifold M also admits a definite HKT metric generating a $D(2, 1; \alpha')$-symmetry.
(2) If $\alpha < 0$ and $\alpha \neq -1$ then M also admits a definite HKT metric with $D(2, 1; \alpha')$-symmetry for each $\alpha' < 0$.

Proof. By rescaling we may choose the special homothety X to have type $(a, 1)$. By replacing g by $-g$ if necessary, we may also ensure that $\mu > 0$. The transformations $\mu \to \mu^k$ give the desired metrics for $\alpha' \neq -1$. The case $\alpha' = -1$ is obtained as $g_{\log\mu}$ when $\alpha < 0$ and $\alpha \neq -1$. □

Remark 3.3. Locally, one may change the parameter in a $D(2, 1; -1)$-symmetry only if $X \lrcorner c = 0$. In this situation the distribution orthogonal to X is integrable and we may locally solve the equation $d\mu = \mu X^\flat$ to obtain an HKT potential μ. This may be used to form g_{μ^2} which has X as a special homothety of type $(g(X, X), b)$.

4. The QKT Quotient

Let us first define what is meant by a QKT structure on a manifold N of dimension $4n$.

The data consists of a metric g, a connection ∇^N and subbundle $\mathcal{G}$ of $\operatorname{End} TN$. The bundle $\mathcal{G}$ should locally have a linear basis $\{I_N, J_N, K_N\}$ satisfying the quaternion relations $I_N^2 = J_N^2 = K_N^2 = -1$ and $I_N J_N = K_N = -J_N I_N$. Call such a triple $\{I_N, J_N, K_N\}$ a *quaternion basis* for $\mathcal{G}$. The metric g is required to be Hermitian with respect to each of these basis elements I_N, J_N and K_N. The pair $(g, \mathcal{G})$ thus constitutes an almost quaternion Hermitian manifold.

The connection ∇^N should be metric, $\nabla^N g = 0$, and quaternionic, so $\nabla^N I_N$ is a linear combination of J_N and K_N. In addition its torsion tensor c^N should be totally skew and of type $(2, 1) + (1, 2)$ with respect to each I_N. If these conditions are satisfied then $(g, \mathcal{G}, \nabla^N)$ is called a QKT structure.

Note that the type condition is the same as saying that the torsion T^N satisfies the relation (2.1) for each choice of $I = I_N$.

Lemma 4.1. *If X is a special homothety of type (a, b) with $a, b \neq 0$, then X, IX, JX, KX generate a local action of $\mathbb{H}^*$.*

Proof. We compute the Lie brackets. First,

$$\begin{aligned}[X, IX] &= \nabla_X(IX) - \nabla_{IX}X - T(X, IX)\\ &= \tfrac{1}{2}aIX - \tfrac{1}{2}aIX = 0,\end{aligned}$$

so X is central. For the remainder we have

$$\begin{aligned}[IX, JX] &= \nabla_{IX}JX - \nabla_{JX}IX - T(IX, JX)\\ &= -aKX - T(IX, JX).\end{aligned}$$

However, Eq. (2.3) gives

$$\begin{aligned}c(IX, JX, Z) &= -c(IX, X, JZ) - (a + b)F_I(JX, Z)\\ &= -(a + b)(KX)^\flat(Z),\end{aligned}$$

and hence $[IX, JX] = bKX$. □

Remark 4.2. When $a = 0$, one can show that X, IX, JX, KX generate a local action of $\mathbb{H}^*$ if $X \lrcorner c = 0$. In this situation one has $\nabla X = 0$, so $[X, IX] = 0$ if and only if $X \lrcorner c = 0$. The relation $[IX, JX] = -b(KX)^\flat$ follows from (2.3).

Theorem 4.3. *Let M^{4m} be an* HKT *manifold with a special homothety X of type (a, b) with $a, b, 0$ unequal and potential μ from Theorem 2.2. Suppose that the vector fields IX, JX, KX are complete and let $Sp(1)$ be the corresponding subgroup of $\mathbb{H}^*$. For $x \neq 0$, the group $Sp(1)$ acts semi-freely on $\mu^{-1}(x)$ and the quotient $\mu^{-1}(x)/\,Sp(1)$ is a* QKT *orbifold.*

The proof will occupy the rest of this section. First note that as $\mu = 2g(X, X)/(a(a-b))$ and $d\mu = 2X^\flat/(a-b)$, each $x \neq 0$ is a regular value of μ and X is not null on $\mu^{-1}(x)$. The group $Sp(1)$ acts semi-freely on

$$S = \mu^{-1}(x),$$

since IX preserves g and commutes with X. The action of $Sp(1)$ is isometric by (2.2b), so we get a Riemannian metric g_N on the quotient $N = S/\,Sp(1)$. Let $\pi\colon S \to N$ be the projection and write $i\colon S \to M$ for the inclusion.

We define local almost complex structures I_N, J_N and K_N on N as follows. Since $\ker \pi_*$ is spanned by IX, JX and KX, the horizontal distribution $\mathcal{H} = (\ker \pi_*)^\perp \subset TS$ is of dimension $4n$, where $n = m - 1$ and is preserved by I, J and K. Thus each point $s \in \pi^{-1}(p)$ defines a triple I_N, J_N, K_N of almost complex structures on $T_pN \cong \mathcal{H}_s$. If s' is another point of $\pi^{-1}(p)$, then $s' = gs$ for some $g \in Sp(1)$. But the action of g permutes I, J and K, so the almost complex structures I'_N, J'_N, K'_N determined by s' are linear combinations of I_N, J_N and K_N. The metric g_N is Hermitian with respect to each of these almost complex structures.

In order to construct a QKT structure on N we need a connection ∇^N. On M we have $\nabla = \nabla^{\mathrm{LC}} + \frac{1}{2}T$, where ∇^{LC} is the Levi-Civita connection of g. This equation is also valid on S, since Lemma 2.1 says that $X \lrcorner c = 0$, so T has no component normal to S. In particular, S with the induced metric naturally carries a metric connection whose torsion is skew.

Now on S, the $Sp(1)$-action is isometric, and so preserves the Levi-Civita connection. On the other hand the torsion T is $Sp(1)$-invariant. To see this, first note that for a two-form β we have

$$\begin{aligned} L_{IX}(J\beta)(Y, Z) &= (IX)(\beta(JY, JZ)) - \beta(J[IX, Y], JZ) - \beta(JY, J[IX, Z]) \\ &= (IX)(\beta(JY, JZ)) - \beta([IX, JY] - bKY, JZ) \\ &\quad - \beta(JY, [IX, JZ] - bKZ) \\ &= J(L_{IX}\beta)(Y, Z) + b\beta(KY, JZ) + b\beta(JY, KZ). \end{aligned}$$

If β is of type $(1, 1)_I$ then this simplifies to $L_{IX}(J\beta) = JL_{IX}\beta$. Taking $\beta = dId\mu$, we now have

$$\begin{aligned} L_{IX}c &= -\tfrac{1}{2}L_{IX}d_Id_Jd_K\mu = \tfrac{1}{2}d_IL_{IX}JdId\mu \\ &= -\tfrac{1}{2}d_Id_Jd_KL_{IX}\mu = 0. \end{aligned}$$

We define ∇^N by

$$\nabla^N_A B = \pi_* \nabla_{\tilde{A}} \tilde{B},$$

where $\tilde{A}$ and $\tilde{B}$ are the $Sp(1)$-invariant lifts of A and B on N to $\mathcal{H} \subset TS$. By the above discussion, we have

$$\nabla^N_A B = \nabla^{\mathrm{LC},N}_A B + \xi_A B,$$

where $\xi_A B = \frac{1}{2}\pi_* T(\tilde{A}, \tilde{B})$ and $\nabla^{\mathrm{LC},N}$ is the Levi-Civita connection of g^N. Since the torsion T is $Sp(1)$-invariant, we see that ∇^N is well-defined. Also the torsion three-tensor is

$$c^N(A, B, C) = c(\tilde{A}, \tilde{B}, \tilde{C}),$$

so c^N is skew symmetric and of type $(2, 1)+(1, 2)$ for each almost complex structure I_N.

Finally, we need to check that ∇^N preserves the metric and almost complex structures. As $I_N A = \pi_*((rI + sJ + tK)\tilde{A})$ for some functions r, s and t on S, we have

$$\begin{aligned}(\nabla^N_A I_N)(B) &= \pi_* \left[\nabla_{\tilde{A}}(rI + sJ + tK)\tilde{B} - (rI + sJ + tK)\nabla_{\tilde{A}}\tilde{B}\right] \\ &= \pi_*((\tilde{A}r)I + (\tilde{A}s)J + (\tilde{A}t)K)\tilde{B},\end{aligned}$$

which is a linear combination of $I_N B$, $J_N B$ and $K_N B$. Thus ∇^N is quaternionic. Looking at g^N, we get

$$\begin{aligned}\pi^*(\nabla^N g^N)(A, B, C) &= \pi^*(Ag^N(B, C) - g^N(\nabla^N_A B, C) \\ &\qquad - g^N(B, \nabla^N_A C)) \\ &= (\nabla g)(\tilde{A}, \tilde{B}, \tilde{C}) = 0,\end{aligned}$$

so ∇^N is metric.

As the c^N has type $(2, 1) + (1, 2)$ for each I_N, we see that N is QKT. □

5. The Geometry of the Quotient

Let N be the QKT quotient constructed in the previous section. Here we will investigate special properties of this manifold. First note that if the HKT metric has signature $(4p, 4q)$ then the metric on N has signature $(4p-4, 4q)$ if $g(X, X) > 0$ or $(4p, 4q-4)$ if $g(X, X) < 0$. In particular, the metric on N may be definite even if the original HKT metric on M is not.

Since ∇^N is a quaternionic connection we have that the curvature satisfies

$$R^N_{A,B} I_N = -\beta_K(A, B)J_N + \beta_J(A, B)K_N, \quad \text{etc.} \tag{5.1}$$

for some two-forms β_I, β_J and β_K.

Proposition 5.1. *β_I is of type $(1, 1)$ with respect to I_N.*

Proof. Write $A = \pi_*\tilde{A}$ and $I_N A = \pi_*((rI + sJ + tK)\tilde{A})$ as the push-forward of invariant vector fields on S. Then $rI + sJ + tK$ is invariant under the action of IX, JX and KX, and we have

$$\begin{aligned}0 &= L_{IX}(rI + sJ + tK) \\ &= ((IX)r)I + ((IX)s)J + ((IX)t)K + bsK - btJ,\end{aligned}$$

so $(IX)r = 0$, $(IX)s = bt$ and $(IX)t = -bs$. Further such relations are obtained by considering the Lie derivative with respect to JX and KX.

The curvature $R^N_{A,B} I_N$ pulls-back to

$$\begin{aligned}
&([\tilde{A}, \tilde{B}]^{\mathcal{V}} r)I + ([\tilde{A}, \tilde{B}]^{\mathcal{V}} s)J + ([\tilde{A}, \tilde{B}]^{\mathcal{V}} t)K \\
&\quad = \frac{1}{g(X,X)} \Big\{ \Big(g([\tilde{A}, \tilde{B}], IX)((IX)r) + g([\tilde{A}, \tilde{B}], JX)((JX)r) \\
&\qquad\qquad + g([\tilde{A}, \tilde{B}], KX)((KX)r) \Big) I + \text{etc.} \Big\} \\
&\quad = \frac{b}{g(X,X)} \Big\{ -g([\tilde{A}, \tilde{B}], JX)tI + g([\tilde{A}, \tilde{B}], KX)sI \\
&\qquad\qquad + g([\tilde{A}, \tilde{B}], IX)tJ - g([\tilde{A}, \tilde{B}], KX)rJ \\
&\qquad\qquad - g([\tilde{A}, \tilde{B}], IX)sK + g([\tilde{A}, \tilde{B}], JX)rK \Big\}.
\end{aligned}$$

Evaluating at $(r, s, t) = (1, 0, 0)$ we see that β_J is given by

$$\begin{aligned}
g([\tilde{A}, \tilde{B}], JX) &= g(\nabla_{\tilde{A}} \tilde{B} - \nabla_{\tilde{B}} \tilde{A} - T(\tilde{A}, \tilde{B}), JX) \\
&= \tilde{A} g(\tilde{B}, JX) - g(\tilde{B}, \nabla_{\tilde{A}} JX) \\
&\quad - \tilde{B} g(\tilde{A}, JX) + g(\tilde{A}, \nabla_{\tilde{B}} JX) \\
&\quad - c(\tilde{A}, \tilde{B}, JX) \\
&= -aF_J(\tilde{A}, \tilde{B}) - c(\tilde{A}, \tilde{B}, JX) \\
&= \frac{b-a}{2} dJd\mu(\tilde{A}, \tilde{B}),
\end{aligned}$$

which is of type $(1, 1)_J$. $\square$

To understand this curvature better we need to consider the relationship of the torsion connection with the underlying quaternionic geometry.

First recall that HKT structures are built on top of hypercomplex structures. In [10] Obata showed that a hypercomplex manifold admits a unique torsion-free connection ∇^{Ob} preserving the complex structures. Similarly, a QKT manifold admits torsion-free connections preserving the quaternionic structure; these are no longer unique but form an affine space modelled on the one-forms.

Lemma 5.2. *Suppose $(Q, \mathcal{G})$ is an almost quaternionic manifold and that $T \in \Gamma(\Lambda^2 T^*Q \otimes TQ)$ is a tensor satisfying Eq.* (2.1) *with respect to almost complex structures I, J, K forming a quaternion basis of $\mathcal{G}$. Define $\xi \in \Gamma T^*Q \otimes \mathrm{End}(TQ)$ by*

$$\begin{aligned}
\xi_Y Z = &-\tfrac{1}{2} T(Y, Z) + \tfrac{1}{6}\big(IT(Y, IZ) + JT(Y, JZ) + KT(Y, KZ) \\
&\qquad - IT(IY, Z) - JT(JY, Z) - KT(KY, Z) \big) \\
&- \tfrac{1}{12}\big(IT(JY, KZ) + JT(KY, IZ) + KT(IY, JZ) \\
&\qquad - IT(KY, JZ) - JT(IY, KZ) - KT(JY, IZ) \big).
\end{aligned}$$

Then ξ satisfies

(1) $\xi_Y Z - \xi_Z Y = T(Y, Z)$,
(2) $\xi I = 0$,
(3) ξ is independent of the choice of quaternion basis $\{I, J, K\}$ for $\mathcal{G}$. $\square$

The proof of the lemma and the following proposition are straightforward computations.

Proposition 5.3. *Let $(N, \mathcal{G}, \nabla)$ be a QKT manifold. Set*

$$\nabla^q = \nabla + \xi,$$

where ξ is given by Lemma 5.2. Then ∇^q is a torsion-free quaternionic connection on N with $\nabla^q A = \nabla A$, for each local section A of $\mathcal{G}$. $\square$

The uniqueness of the Obata connection now gives:

Corollary 5.4. *Let (M, ∇) be an HKT manifold. Then the Obata connection is given by $\nabla^{Ob} = \nabla + \xi$ with ξ defined in Lemma 5.2.* $\square$

The above proposition is useful as it allows us to apply information about quaternionic curvature from the work of Alekseevsky and Marchiafava [1] to give an interpretation of Proposition 5.1.

First consider the case when $\dim N > 4$. As ∇^{q} is a quaternionic connection, its curvature R^{q} may be written as

$$R^{\mathrm{q}} = R^B + W^{\mathrm{q}}, \tag{5.2}$$

where W^{q} is an algebraic curvature tensor for $\mathfrak{sl}(n, \mathbb{H})$ and R^B is determined by an element $B \in \Gamma(T^*N \otimes T^*N)$. The component W^{q} is independent of the choice of the torsion-free quaternionic connection and acts trivially both on $\mathcal{G}$ and the real canonical bundle $\kappa^{\mathbb{R}} = \Lambda^{4n} T^*N$. The curvature of $\mathcal{G}$ is

$$\beta_I(Y, Z) = 2\big(B(Y, I_N Z) - B(Z, I_N Y)\big).$$

Write $2B = \lambda_I^{1,1} + \lambda_I^{2,0} + \sigma_I^{1,1} + \sigma_I^{2,0}$ according to the splittings of $\Lambda^2 T^*N$ and $S^2 T^*N$ with respect to I. Here $\lambda_I^{2,0}$ denotes the component of the skew-symmetric part of $2B$ in $\Lambda_I^{2,0} N + \Lambda_I^{0,2} N$, and similar notation is used for σ. We have $\beta_I = \sigma_I^{1,1}(\cdot, I\cdot) + \lambda_I^{2,0}(\cdot, I\cdot)$. Proposition 5.1 thus implies that $\lambda_I^{2,0} = 0$ for all I, i.e., the skew-part of B is of type $(1, 1)$ for all I. However, the skew-part of B is the curvature of ∇^q on $\kappa^{\mathbb{R}}$. Thus the curvature of $\kappa^{\mathbb{R}}$ is of type $(1, 1)$ for all I. In other words ∇^{q} induces a instanton connection on $\kappa^{\mathbb{R}}$.

For N a four-manifold, the decomposition (5.2) has an extra term W_-: the anti-self-dual part of the Weyl curvature. If we assume $W_- = 0$, then the above analysis applies also in dimension four.

Definition 5.5. *Let N be a QKT manifold. If* $\dim N = 4$ *suppose in addition that N is self-dual. We say that N is* of instanton type *if ∇^{q} induces an instanton connection on the real canonical bundle $\kappa^{\mathbb{R}}$.*

Remark 5.6. Comparing with quaternionic or quaternionic Kähler geometry it would have been natural to include self-duality in the definition of QKT manifolds for dimension four. However, that goes against the established definitions in the QKT literature.

Note that the above discussion implies that QKT manifolds of instanton type are precisely those for which the curvature forms β_I are of type $(1,1)_I$. We summarise the above discussion in the following result.

Theorem 5.7. *Let N be a QKT manifold which is an $\mathbb{H}^*$-quotient of an HKT manifold as in Theorem 4.3. Then N is of instanton type.* □

This condition may be related to the torsion of the QKT manifold as follows. Recall that there is a torsion *one-form* τ given by

$$\tau(A) = \tfrac{1}{2}\sum_{i=1}^{4n} c^N(IA, e_i, Ie_i),$$

where $\{e_i\}$ is an orthonormal basis for TN. This one-form is independent of I and globally defined [7].

Proposition 5.8. *Let N be a QKT manifold. If* $\dim N = 4$, *suppose also that N is self-dual. Then N is of instanton type if and only if its torsion one-form τ satisfies $d\tau \in \bigcap_I \Lambda_I^{1,1}$.*

Proof. As ∇ is a metric connection, we have that $\nabla^{q}\mathrm{vol}^N = \xi\cdot\mathrm{vol}^N = \sum_{i=1}^{4n} g^N(\xi e_i, e_i)\,\mathrm{vol}^N$. Let $\{f_1,\dots,f_n\}$ be an orthonormal quaternionic basis for TN. Then

$$\begin{aligned}
&\sum_{i=1}^{4n} g^N(\xi_A e_i, e_i)\\
&\quad= 4\sum_{j=1}^{n} g^N(\xi_A f_j, f_j)\\
&\quad= 4\sum_{j=1}^{n} \tfrac{1}{6}\big(-c^N(IA, f_j, If_j) - c^N(JA, f_j, Jf_j) - c^N(KA, f_j, Kf_j)\big)\\
&\qquad - \tfrac{1}{12}\big(c^N(JA, Kf_j, If_j) + c^N(KA, If_j, Jf_j) + c^N(IA, Jf_j, Kf_j)\\
&\qquad - c^N(KA, Jf_j, If_j) - c^N(IA, Kf_j, Jf_j) - c^N(JA, If_j, Kf_j)\big)\\
&\quad= -\tfrac{1}{6}\sum_{j=1}^{n} c^N(IA, f_j, If_j) + c^N(IA, Jf_j, Kf_j) + c^N(JA, f_j, Jf_j)\\
&\qquad + c^N(JA, Kf_j, If_j) + c^N(KA, f_j, Kf_j) + c^N(KA, If_j, Jf_j)\\
&\quad= -\tfrac{1}{4}\tau(A).
\end{aligned}$$

Thus the curvature of $\kappa^{\mathbb{R}}$ is $-\frac{1}{4}d\tau$, giving the result. □

6. An Inverse Construction

Let N^{4n} be a QKT manifold of instanton type. We follow the constructions and notation of [13]. Set $\mathcal{U}(N)$ to be the bundle

$$\mathcal{U}(N) = P \times_{Sp(n)\,Sp(1)} \mathbb{H}^*/\{\pm 1\},$$

where P is the principal $Sp(n)\,Sp(1)$-bundle of frames over N. This is a real line bundle over the bundle $\mathcal{S}(N)$ of quaternion bases for $\mathcal{G}$. Let ω be the connection one-form on P defined by the torsion-connection ∇. As ∇ is an $Sp(n)\,Sp(1)$-connection, ω takes values in $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$. Let ω_- be the $\mathfrak{sp}(1) \cong \operatorname{Im}\mathbb{H}$ part of ω. The form ω_- defines a splitting of $T\,\mathcal{U}(N) = \mathcal{H} + \mathcal{V}$, with $\mathcal{H} \cong T_pN$ and $\mathcal{V} \cong \mathbb{H}$. The projection to $\mathcal{S}(N)$ defines three almost complex structures on $\mathcal{H}$. Combining these with those on $\mathcal{V}$ one gets a hypercomplex structure on $\mathcal{U}(N)$. This follows from the results of [11], since Proposition 5.3 shows that ω_- extends to a torsion-free quaternionic connection and $\mathcal{U}(N)$ is $\mathcal{U}_{(n-1)/n}(N)$ twisted by the instanton bundle $(\kappa^{\mathbb{R}})^{-1/4(n+1)}$.

Consider the function

$$\mu = x\bar{x}$$

on $\mathcal{U}(N)$, where x is the quaternionic coordinate on $\mathbb{H}^*$. Then $d\mu = x\bar{\psi} + \psi\bar{x}$, where $\psi = dx - x\omega_-$, and so $Id\mu = -x\bar{\psi}i + i\psi\bar{x}$. Thus we have

$$dId\mu = -\psi \wedge \bar{\psi}i - i\psi \wedge \bar{\psi} - x\Omega_-\bar{x}i - ix\Omega_-\bar{x},$$

where $\Omega_- = d\omega_- + \omega_- \wedge \omega_- = -\frac{1}{2}(\beta_I i + \beta_J j + \beta_K k)$ is the $\mathfrak{sp}(1)$-curvature.

The function μ is an HKT potential if $F_I = \frac{1}{2}(dId - JdId)\mu$ defines a Kähler form for I associated to a non-degenerate metric via $g(I\cdot,\cdot) = F_I(\cdot,\cdot)$. Vertically, F_I is given by the ψ terms and is the Kähler form of the flat structure on $\mathbb{H}^*$.

Horizontally, we get a condition on Ω_-. At $x = 1$ the horizontal part of F_I is $\frac{1}{2}(\beta_I - J\beta_I)$. As N is of instanton type we have $\beta_I = \sigma_I^{1,1}(\cdot, I\cdot)$, in the notation of the previous section. This gives that the horizontal part of g at $x = 1$ is $-\frac{1}{2}(1+J_N)\sigma_I^{1,1} = \sigma^q$, where σ^q is the component of the symmetric part of $-2B$ that is of type $(1, 1)$ for each I_N. For general x, the horizontal part of g is now $x\bar{x}\sigma^q$. We call $\sigma^q = \frac{1}{2}(\beta_I - J_N\beta_I)(\cdot, I_N\cdot)$ the *curvature metric* of N. Recall that β_I is defined by Eq. (5.1) and so we have

$$\sigma^q = \frac{1}{4n}\sum_{i=1}^{4n} R^N(X, I_NY, e_i, I_Ne_i) + R^N(J_NX, K_NY, e_i, I_Ne_i).$$

The group $\mathbb{H}^*$ acts on $\mathcal{U}(N)$ via left-multiplication and generates a special homothety of type $(2, -2)$, i.e., we have a $D(2, 1; -2)$-symmetry. Using the results of Sect. 3 we now obtain HKT structures with $D(2, 1; \alpha)$-symmetry for other values of α.

Theorem 6.1. *Let N be a QKT manifold of instanton type whose curvature metric is non-degenerate of signature $(4p, 4q)$. Then $\mathcal{U}(N)$ carries HKT structures with $D(2, 1; \alpha)$-symmetry for each $\alpha \neq 0$. For $\alpha < 0$ the metric has signature $(4p + 4, 4q)$; for $\alpha > 0$ the signature is $(4q + 4, 4p)$.* □

Example 6.2. If N is a quaternionic Kähler manifold with non-zero scalar curvature s, then the curvature metric is $\sigma^q = s'g^N$, where s' is a positive dimension-dependent multiple of s. The HKT structure on $\mathcal{U}(N)$ is thus positive definite exactly when $s\alpha < 0$. In [13] it was shown that the metric constructed from $\mu = x\bar{x}$ is hyperKähler with $D(2, 1; -2)$-symmetry. For $N = \mathbb{HP}(n)$, we have $\mathcal{U}(N) = \mathbb{H}^{n+1}/\{\pm 1\}$ with the flat hyperKähler metric. For $N = \mathbb{H}\mathrm{H}(n)$, or any other quaternionic Kähler manifold of negative scalar curvature, the hyperKähler metric on $\mathcal{U}(N)$ is indefinite; but Theorem 6.1 above shows that $\mathcal{U}(N)$ has a definite HKT metric with $D(2, 1; 1)$-symmetry.

Remark 6.3. Suppose N is a QKT manifold constructed as the $\mathbb{H}^*$-quotient of an HKT manifold with $D(2, 1; \alpha)$-symmetry as in Theorem 4.3. The proof of Proposition 5.1 shows that σ^q is determined by the HKT potential and is the horizontal part of g up to a constant. But that is also how g^N is constructed. So N has curvature metric σ^q proportional to g^N and the construction in this section is inverse to that of Theorem 4.3.

Remark 6.4. In [6] a construction similar to Theorem 6.1 is given, with different assumptions on the base and with the conclusion that $\mathcal{U}(N)$ is hyperKähler. The essential condition in [6] is that dc^N should be of type (2, 2) for each I. The appendix of that paper contains a proof that this implies that the curvature metric is quaternionic Kähler.

Ivanov [7] showed that every metric $\bar{g}^N = e^u g^N$ conformal to a QKT metric g^N admits a QKT connection $\bar{\nabla}^N$. The torsion-one form satisfies $\bar{\tau} = \tau - (2n+1)du$, so $(\bar{g}^N, \bar{\nabla}^N)$ is of instanton type if and only if (g^N, ∇^N) is. One computes that

$$\begin{aligned}\bar{\beta}_I(X, Y) = &\beta_I(X, Y) + (\nabla^q du)(IX, Y) - (\nabla^q du)(IY, X)\\ &- (Jdu \wedge Kdu)(X, Y),\end{aligned}$$

and hence

$$\bar{\sigma}^q = \sigma^q + \tfrac{1}{2}(1 + I + J + K)\nabla^q du + \tfrac{1}{2}(d^{\mathbb{H}}u)^2. \tag{6.1}$$

We may use this result in several ways. Firstly, note that it shows that for a general QKT manifold N of instanton type, the curvature metric σ^q need not be proportional to g^N. Secondly, if g^N has degenerate curvature metric, then in a neighbourhood of any point we can choose u so that $\bar{\sigma}^q$ in (6.1) is non-degenerate.

Proposition 6.5. *Suppose N is a quaternionic manifold. Then locally, N admits a positive definite QKT structure.*

Proof. Fix a volume form vol_0 on N. Then there is a unique torsion-free quaternionic connection ∇^0 on N such that $\nabla^0\mathrm{vol}_0 = 0$. If the curvature metric σ_0 is not positive definite, replace vol_0 be $e^{2nu}\mathrm{vol}_0$ for some function u; then σ_0 will change as in (6.1) and we may choose u so that the curvature metric is positive definite in a neighbourhood of a given point. Now set $g^N = \sigma^0$. Since ∇^0 preserves vol_0 the bundle $\kappa^{\mathbb{R}}$ is an instanton and we may use g^N and ∇^0 to construct an HKT structure on $\mathcal{U}(N)$ as in Theorem 6.1. By Theorem 4.3 we get a QKT structure on N which by Remark 6.3 has g^N as its metric, up to a constant scale. □

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