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Neutral Calabi-Yau Structures on Kodaira Manifolds

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Abstract: We construct neutral Calabi-Yau metrics and hypersymplectic structures on some Kodaira manifolds. Our structures are symmetric with respect to the central tori.

1. Introduction

This paper investigates solutions of the Einstein equations in dimension $4n$ with signature $(2n, 2n)$. Our solutions are obtained by constructing neutral Calabi-Yau metrics fibered by Lagrangian tori. Some of the solutions are in fact hypersymplectic and such structures play a role in string theory (cf. [16]). Indeed, the target manifolds of sigma models with twisted (4,1)-superconformal symmetries are hypersymplectic.

In [17] Kamada studied hypersymplectic structures on compact manifolds of dimension 4 and he proved that any complex surface with a hypersymplectic structure must be biholomorphic either to a complex torus or a primary Kodaira surface. The latter surfaces are elliptic fibrations over elliptic curves with $b_1 = 3$ and they can be considered as the compact quotient of $\mathbb{C}^2$ by a discrete subgroup of the affine group of $\mathbb{C}^2$ [1, 18]. A special type of primary Kodaira surface is the Kodaira-Thurston surface [22], which is a compact quotient of the 2-step nilpotent Lie group $H_3 \times \mathbb{R}$ with 2-dimensional center (H_3 is the real 3-dimensional Heinsenberg group) by a uniform discrete subgroup. A primary Kodaira surface cannot have a positive definite Kähler metric, since its first Betti number is three. However, the Kodaira-Thurston surface does support neutral Kähler metrics of signature (2, 2) (see [17, 11]) and, in particular, a symplectic form [22].

Let G be a simply-connected real 2-step nilpotent Lie group of dimension $4n$, with center Z of dimension $2n$. Assume that the Lie algebra $\mathfrak{g}$ has an integrable almost complex structure which preserves the center $\mathfrak{z}$ of $\mathfrak{g}$. Generalizing the notion of Kodaira-Thurston

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surface, we define a *Kodaira manifold* (M, I) as a compact quotient $M = \Gamma\backslash G$ by a uniform discrete subgroup endowed with the invariant complex structure I associated to the complex structure of $\mathfrak{g}$.

The nilpotency of G is equivalent to the existence of a basis $\{e^1, \dots, e^{4n}\}$ of left-invariant 1-forms on G such that

$$de^i \in \Lambda^2\langle e^1, \dots, e^{i-1}\rangle, \qquad 1 \le i \le 4n,$$

where the right-hand side is interpreted as zero for $i = 1$ [21]. For any such G with rational structure constants, there is a discrete subgroup Γ such that $M = \Gamma\backslash G$ is a compact manifold [19], known as a compact nilmanifold. By [20] M is the total space of a principal torus bundle over a torus. More precisely, the center of G is a rational subgroup of G and we have a principal torus bundle $\pi : M \to T^{2n}$, with fiber the central torus $T^Z = (\Gamma \cap Z)\backslash Z$.

When $n = 1$, a (non-abelian) nilpotent Lie group admitting an invariant complex structure is isomorphic to the 2-step nilpotent Lie group $H_3 \times \mathbb{R}$ [7].

It is well known that a compact nilmanifold cannot be Kähler, unless it is a torus [10, 2, 13, 6]. Then a Kodaira manifold cannot support (positive definite) Kähler metrics.

We recall that a neutral Kähler manifold is a complex manifold (M^{4n}, I) with a neutral metric g (i.e., of signature $(2n, 2n)$) such that $g(IX, IY) = g(X, Y)$, for any tangent vector field X, Y of M and I is parallel with respect to the Levi-Civita connection ∇.

Such a manifold is said to be a *neutral Calabi-Yau* if there exists $\Phi \in \Lambda^{2n,0}T^*M$ such that $\Phi \neq 0$ and $\nabla\Phi = 0$. It is easily seen that a neutral Calabi-Yau manifold is Ricci-flat (see [15] for the positive case).

Special examples of neutral Calabi-Yau manifolds are given by *hypersymplectic* manifolds. Following Hitchin's definition [14], a hypersymplectic structure on a $4n$-dimensional manifold M is given by (I, J, K, g), where g is a neutral metric and I, J, K are endomorphisms of the tangent bundle such that

$$\begin{aligned} &I^2 = -\mathrm{Id},\ J^2 = K^2 = \mathrm{Id},\ IJ = -JI = K, \\ &g(X, Y) = g(IX, IY) = -g(JX, JY) = -g(KX, KY), \end{aligned}$$

for all vector fields X, Y on M, whose corresponding fundamental forms $\Omega_E(X, Y) = g(EX, Y)$, $E \in \{I, J, K\}$, are closed. Using the fact that I, J, K are parallel with respect to the Levi-Civita connection ∇, it is easy to check that in addition the form $\Phi = (\Omega_J + i\Omega_K)^n \in \Lambda^{2n,0}T^*M$ is ∇-parallel. Thus hypersymplectic manifolds are neutral Calabi-Yau manifolds.

A submanifold L of a symplectic manifold M is Lagrangian if the symplectic form restricts to zero on L and $\dim M = 2\dim L$. A Lagrangian submanifold of a neutral Calabi-Yau manifold is called *special Lagrangian* if in addition the real part of Φ vanishes.

In Sect. 2 we prove that any principal T^Z-bundle over T^{2n} admits a neutral Ricci-flat metric and that Kodaira manifolds $\Gamma\backslash G$ with a closed G-invariant non-degenerate (1,1)-form admit T^Z-invariant neutral Calabi-Yau metrics with special Lagrangian tori.

In Sect. 3 we give examples of 8-dimensional Kodaira manifolds with neutral Calabi-Yau and hypersymplectic structures.

In Sect. 4 we prove the existence of neutral Calabi-Yau structures on compact quotients of cotangent bundles of some 2-step nilpotent Lie groups G. The cotangent bundle T^*G has the Lie group structure considered by Boyom using the Lie group structure of G and a flat torsion-free affine connection on G [3, 4].

In the last section we show that on compact quotients of $T^*(H_3 \times \mathbb{R})$ there exist families of neutral Calabi-Yau structures which contain also hypersymplectic structures.

2. Neutral Calabi-Yau Metrics

In this section we prove that any principal T^Z-bundle over T^{2n} has a neutral Ricci-flat metric and we find a series of new neutral Calabi-Yau solutions on Kodaira manifolds.

Theorem 2.1. *Any principal T^Z-bundle over T^{2n}, with* $\dim T^Z = 2n$*, has a neutral Ricci-flat metric.*

Proof. A principal T^Z-bundle over T^{2n} is the compact quotient of a 2-step nilpotent Lie group G by a uniform discrete subgroup Γ. Since the Lie algebra $\mathfrak{g}$ is 2-step nilpotent it is possible to choose a basis of left-invariant 1-forms $\{e^1, \dots, e^{4n}\}$ on G such that

(a) $de^i = 0, \qquad i = 1, \dots, 2n,$
(b) $de^j \in \Lambda^2\langle e^1, \dots e^{2n}\rangle, \qquad j = 2n+1, \dots, 4n,$

and the center $\mathfrak{z} = \mathrm{span}\langle e_{2n+1}, \dots, e_{4n}\rangle$ (where $\{e_i\}$ denotes the dual basis of $\{e^i\}$). Let $(x_1, \dots, x_{4n})$ be a system of global coordinates on G such that $e^i = dx_i, i = 1, \dots, 2n$. Define, for $i = 1, \dots, n$,

$$\begin{aligned}
E^{2i} &= e^{2i},\\
E^{2n+2i-1} &= \textstyle\sum_{j=1}^n(-a_{ij}e^{2n+2j-1} - b_{ij}e^{2n+2j} - \frac{c_i}{2}e^{2i}) + \sum_{i<j}^n(-d_{ij}e^{2j-1} - e_{ij}e^{2j}),\\
E^{2n+2i} &= \textstyle\sum_{j=1}^n(-a_{ij}e^{2n+2j} + b_{ij}e^{2n+2j-1} + \frac{c_i}{2}e^{2i-1}) + \sum_{i<j}^n(-d_{ij}e^{2j} + e_{ij}e^{2j-1}),
\end{aligned}$$

with a_{ij} and b_{ij} non-zero constant for any i, j and c_i, d_{ij}, e_{ij} which depend only on the variables $x_1, \dots, x_{2n}$. Thus, $\{E^1, \dots, E^{4n}\}$ is a basis of T^Z-invariant 1-forms on M such that

$$dE^i = 0, \qquad dE^{2n+i} \in C^\infty_{T^{2n}} \otimes \Lambda^2\langle E^1, \dots, E^{2n}\rangle, \quad i = 1, \dots, 2n, \tag{1}$$

where $C^\infty_{T^{2n}}$ are the space of smooth functions on T^{2n}.

Consider the neutral metric g given by

$$\sum_{i=1}^n (E^{2i-1} \cdot E^{2n+2i} - E^{2i} \cdot E^{2n+2i-1}), \tag{2}$$

where $\cdot$ denotes the symmetric product. Using (1), the connection forms θ^i_j of the Levi-Civita connection ∇ associated to g are non-zero only if $i = 2n+1, \dots, 4n$ and $j = 1, \dots, 2n$. Moreover

$$\theta^i_j \in C^\infty_{T^{2n}} \otimes \mathrm{span}\langle E^1, \dots, E^{2n}\rangle. \tag{3}$$

It follows that the curvature 2-forms Ω^i_j satisfy

$$\Omega^i_j = d\theta^i_j + \theta^i_k \wedge \theta^k_j = d\theta^i_j \in C^\infty_{T^{2n}} \otimes \Lambda^2\langle E^1, \dots, E^{2n}\rangle.$$

So each component of the form $R^k_{jk\ell} = R(E_k, E_\ell, E_j, E_k)$ vanishes and, in particular, g is Ricci-flat. $\square$

We begin with a proposition about candidates for neutral Kähler structures on Kodaira manifolds. We seek such structures which are invariant with respect to the central torus action and for which the fibers are special Lagrangian.

We call a 2-form Ω a *special 2-form* on a Kodaira manifold (M, I) of dimension $4n$ if Ω satisfies the following conditions:

1. Ω is non-degenerate;
2. Ω is of type $(1,1)$ with respect to I;
3. Ω is invariant with respect to the action of the central torus T^Z;
4. Ω restricts to zero on the orbits of the central torus T^Z, i.e. $\Omega|_{\pi^{-1}(b)} = 0$, for any b in the base torus T^{2n}.

Proposition 2.1. *Let (M, I) be a Kodaira manifold of dimension $4n$. If Ω is a special 2-form on M then there exists a basis of T^Z-invariant 1-forms*

$$\{E^1, E^2 = IE^1, \ldots, E^{4n-1}, E^{4n} = IE^{4n-1}\}$$

on M such that $\Omega = \sum_{i=1}^{2n} E^{2n+i} \wedge E^i$, with

$$dE^i = 0, i = 1, \ldots, 2n, \quad dE^{2n+i} \in C^\infty_{T^{2n}} \otimes \mathcal{I}(E^1, \ldots, E^{2n}), \qquad i = 1, \ldots, 2n,$$

where $C^\infty_{T^{2n}}$ are the space of smooth functions on T^{2n} and $\mathcal{I}(E^1, \ldots, E^{2n})$ denotes the ideal generated by the set $\{E^1, \ldots, E^{2n}\}$ in the exterior algebra $\Lambda^2\langle E^1, \ldots, E^{4n}\rangle$.

Proof. Since $\mathfrak{g}$ is 2-step nilpotent and I preserves the $2n$-dimensional center $\mathfrak{z}$, it is possible to choose a basis of left-invariant 1-forms $\{e^1, \ldots, e^{4n}\}$ on G which satisfies the conditions (a), (b) and

(c) $e^{2i} = Ie^{2i-1}, \qquad i = 1, \ldots, 2n,$

and the center $\mathfrak{z} = \mathrm{span}\langle e_{2n+1}, \ldots, e_{4n}\rangle$ (where $\{e_i\}$ denotes the dual basis of $\{e^i\}$). Let $(x_1, \ldots, x_{4n})$ be a system of global coordinates on G such that $e^i = dx_i$, $i = 1, \ldots, 2n$. Then a T^Z-invariant $(1,1)$-form Ω vanishing on the fibers is given by

$$\begin{aligned}\Omega = {} & \textstyle\sum_{i,j=1}^n a_{ij}(e^{2i-1} \wedge e^{2n+2j-1} + e^{2i} \wedge e^{2n+2j}) \\ & + \textstyle\sum_{i,j=1}^n b_{ij}(e^{2i-1} \wedge e^{2n+2j} - e^{2i} \wedge e^{2n+2j-1}) \\ & + \textstyle\sum_{i=1}^n c_i e^{2i-1} \wedge e^{2i} \\ & + \textstyle\sum_{1\le i<j\le n} d_{ij}(e^{2i-1} \wedge e^{2j-1} + e^{2i} \wedge e^{2j}) \\ & + \textstyle\sum_{1\le i<j\le n} e_{ij}(e^{2i-1} \wedge e^{2j} - e^{2i} \wedge e^{2j-1}),\end{aligned} \tag{4}$$

where $a_{ij}, b_{ij}, c_i, d_{ij}, e_{ij}$ depend only on the variables $x_1, \ldots, x_{2n}$.

Ω can be rewritten as $\Omega = \sum_{i=1}^n (\alpha^{2i-1} \wedge e^{2i-1} + \alpha^{2i} \wedge e^{2i})$, where

$$\alpha^{2i-1} = \sum_{j=1}^n (-a_{ij}e^{2n+2j-1} - b_{ij}e^{2n+2j} - \frac{c_i}{2}e^{2i}) + \sum_{i<j}^n (-d_{ij}e^{2j-1} - e_{ij}e^{2j})$$

and

$$\alpha^{2i} = \sum_{j=1}^n (-a_{ij}e^{2n+2j} + b_{ij}e^{2n+2j-1} + \frac{c_i}{2}e^{2i-1}) + \sum_{i<j}^n (-d_{ij}e^{2j} + e_{ij}e^{2j-1}).$$

Then by (c) one has that $\alpha^{2i} = I\alpha^{2i-1}$ and, using (a) and (b) we have that $d(\alpha^i) \in C^\infty_{T^{2n}} \otimes \mathcal{I}(e^1, \ldots, e^{2n})$.

If we put $E^i = e^i$ and $E^{2n+i} = \alpha^i$, for $i = 1, \ldots, 2n$, then

$$\begin{aligned}&\Omega = \textstyle\sum_{i=1}^{2n} E^{2n+i} \wedge E^i, \\ &dE^i = 0, \quad dE^{2n+i} \in C^\infty_{T^{2n}} \otimes \mathcal{I}(E^1, \ldots, E^{2n}), \qquad i = 1, \ldots, 2n,\end{aligned}$$

and since Ω is non-degenerate, $\{E^1, \ldots, E^{4n}\}$ is a basis of 1-forms on M. $\square$

A first consequence is the following

Corollary 2.1. *Equation* (4) *defines a special* 2*-form on* M *if and only if* $(a_{ij}, b_{ij})(m) \neq (0, 0)$*, for any* i, j *and for any* $m \in M$.

Setting a_{ij} and b_{ij} to be constant, we obtain

Corollary 2.2. *Let* (M, I) *be a Kodaira manifold of dimension* $4n$*. There exists a special* 2*-form* Ω *such that the basis given by Proposition 2.1 satisfies the conditions* (1).

Proof. Choose Ω as in the proof of Proposition 2.1 but with constant functions a_{ij}, b_{ij}, for any i, j. This is equivalent to $dE^{2n+i} \in C^\infty_{T^{2n}} \otimes \Lambda^2\langle E^1, \dots, E^{2n}\rangle$. □

Theorem 2.2. *Let* (M, I) *be a Kodaira manifold of dimension* $4n$ *and suppose that the type (1,1)-form* Ω *is given as in Corollary 2.2. Then the neutral metric* g *defined by* $g(X, Y) = \Omega(X, IY)$ *is Ricci-flat.*

Proof. Let $p : G \to M = \Gamma\backslash G$. We may write the expression in (4) as

$$\Omega = \Theta + \pi^*\beta, \tag{5}$$

where

$$\Theta = \sum_{i,j=1}^{n} a_{ij}(e^{2i-1}\wedge e^{2n+2j-1} + e^{2i}\wedge e^{2n+2j}) + \sum_{i,j=1}^{n} b_{ij}(e^{2i-1}\wedge e^{2n+2j} - e^{2i}\wedge e^{2n+2j-1}),$$

$p^*\Theta$ is a left-invariant $(1, 1)$-form on G and $\pi^*\beta$ is the pull-back from the base torus.

The neutral metric g corresponding to Ω is given by (2). Using (1) like in the proof of Theorem 2.1 one can prove that g is Ricci-flat. □

Corollary 2.3. *Let* (M, I) *be a Kodaira manifold of dimension* $4n$ *and assume that the non-degenerate* $(1, 1)$*-form* Ω *as in* (5) *is closed, then* (I, g, Ω) *is a neutral Calabi-Yau structure on* M *with the parallel* $(2n, 0)$*-form* Φ *vanishing on the fibers.*

Proof. Since Ω is closed it follows by the previous theorem that the metric $g(X, Y) = \Omega(X, IY)$ is neutral Kähler and Ricci-flat. Moreover, $\Phi = \wedge_{i=1}^{2n}(E^{2i-1} - iE^{2i}) = \wedge_{i=1}^{2n}\phi^i$, is a $(2n, 0)$-form since $\phi^1, \dots, \phi^{2n}$ is a basis of (1,0)-forms. By (3) one has also that

$$\begin{aligned}&\nabla_X E^\ell = 0, \qquad \ell = 1, \dots, 2n,\\ &\nabla_X E^\ell = \textstyle\sum_{k=1}^{2n} \theta^\ell_k E^k, \qquad \ell = 2n+1, \dots, 4n,\end{aligned}$$

and then $\nabla\phi^i$ vanishes for any $i \leq n$ and belongs to the span$\langle\phi^1, \dots, \phi^n\rangle$ for any $i > n$.

Thus $\nabla\Phi = 0$ and since $\Phi = F(x_1, \dots, x_{2n}) \bigwedge_{i=1}^{2n}(e^{2i-1} + ie^{2i})$, where F is a complex function, and the center $\mathfrak{z}$ is spanned by $\langle e_{2n+1}, \dots, e_{4n}\rangle$, the restriction of Φ on the central torus T^Z vanishes. In particular, the fibers are special Lagrangian tori, because the real part of Φ vanishes on the fibers. □

Remark 2.1. Our strategy for finding neutral Kähler forms Ω will be to find closed G-invariant and non-degenerate $(1, 1)$-forms Θ and use the ansatz $\Omega = \Theta + \pi^*\beta$ for any closed $(1, 1)$-form β on the base torus.

Remark 2.2. As a matrix-valued 2-form, the curvature of the neutral Kähler metric is a section of $\mathfrak{so}(2n, 2n) \otimes C^\infty_{T^{2n}} \otimes \Lambda^2\langle E^1, \dots, E^{2n}\rangle$. Due to the Bianchi identity, the holonomy algebra is contained in $\Lambda^2\langle E^1, \dots, E^{2n}\rangle$. This is in accordance with the results in [5].

3. 8-Dimensional Kodaira Manifolds

In this section, we construct several examples of such manifolds. We demonstrate that some of them have non-trivial neutral Calabi-Yau metrics, and some of them have non-trivial hypersymplectic structures.

3.1. Existence of closed $(1, 1)$*-forms.* By (3), if $\Omega = \Theta$ is a closed G-invariant special 2-form then the metric associated to Θ is flat. We will describe a method to construct non-flat neutral Calabi-Yau structures (I, g, Ω) by deforming the flat metric.

Let $(\Gamma\backslash G, I)$ be a Kodaira manifold of dimension 8. Then there exists a basis of left-invariant 1-forms $\{e^1, \dots, e^8\}$ on G which satisfies the conditions (a), (b), (c) (see proof of Proposition 2.1) and a G-invariant closed special 2-form. Let

$$\pi^*\beta = h_1 e^1 \wedge e^2 + h_2 e^3 \wedge e^4 + h_3(e^1 \wedge e^4 - e^2 \wedge e^3) - h_4(e^1 \wedge e^3 + e^2 \wedge e^4)$$

be a pull-back of a $(1, 1)$-form on the base torus, where $h_i(x_1, x_2, x_3, x_4)$, for $i = 1, \dots, 4$ are the pull-back of functions on the base T^4 and $e^i = dx_i$, for $i = 1, \dots, 4$. Then the 2-form

$$\begin{aligned}\Omega = \Theta &+ h_1 e^1 \wedge e^2 + h_2 e^3 \wedge e^4 \\ &+ h_3(e^1 \wedge e^4 - e^2 \wedge e^3) - h_4(e^1 \wedge e^3 + e^2 \wedge e^4)\end{aligned} \tag{6}$$

is still a special 2-form on M and it is closed if and only if the functions h_i satisfy the following conditions

$$\begin{cases} (h_1)_{x_3} - (h_3)_{x_1} + (h_4)_{x_2} = 0, & (h_1)_{x_4} - (h_3)_{x_2} - (h_4)_{x_1} = 0, \\ (h_2)_{x_1} - (h_3)_{x_3} - (h_4)_{x_4} = 0, & (h_2)_{x_2} - (h_3)_{x_4} + (h_4)_{x_3} = 0, \end{cases} \tag{7}$$

where $(h_i)_{x_j}$ denotes the partial derivative of h_i with respect to x_j. By Corollary 2.3, (I, g, Ω) is a neutral Calabi-Yau structure on M if and only if h_i are solutions of Eq. (7). For instance $h_1 = h_1(x_1, x_2)$, $h_2 = h_2(x_3, x_4)$ and h_3, h_4 constants is always a solution.

Other solutions may be obtained by means of Fourier series. Consider the integrability condition

$$(h_3)_{x_1 x_4} - (h_3)_{x_2 x_3} = (h_4)_{x_1 x_3} + (h_4)_{x_2 x_4},$$

substitute Fouries series

$$h_k(x_1, x_2, x_3, x_4) = \sum A^k_{n_1 n_2 n_3 n_4} e^{i(n_1 x_1 + n_2 x_2 + n_3 x_3 + n_4 x_4)},$$

$k = 1, \dots, 4$, and solve for h_3, h_1 and h_2 given h_4. For example

$$\begin{aligned} h_1 &= h_2 = \tfrac{5}{3}\mathcal{R}e(e^{i(2x_1 + x_2 + x_3 + 2x_4)}), \\ h_3 &= \tfrac{4}{3}\mathcal{R}e(e^{i(2x_1 + x_2 + x_3 + 2x_4)}) \\ h_4 &= \mathcal{R}e(e^{i(2x_1 + x_2 + x_3 + 2x_4)}) \end{aligned}$$

is a solution.

3.2. Examples. We are going to focus on two types of 2-step nilpotent Lie groups. One of them appears as the underlying manifolds for an invariant hypercomplex structure [8]. Another appears as 8-dimensional central extensions of 6-dimensional 2-step nilpotent groups endowed with an invariant abelian complex structure [21].

Note that if a Kodaira manifold admits a G-invariant hypersymplectic structure, the corresponding metric is flat. As in the last section, we shall keep the complex structure of the underlying Kodaira manifold, and deform the (1,1)-form so that we have a new non-flat hypersymplectic structure. We shall deal with four examples.

Example 1. In this and the next construction, we consider examples coming from hypercomplex nilmanifolds [8]. We shall focus on constructions of hypersymplectic structures.

The first one is the real 8-dimensional central extension G_1 of the complex 3-dimensional Heisenberg group. Its structural equations are

$$de^i = 0, \quad i = 1, \dots, 6, \text{ and } \quad de^7 = e^1 \wedge e^3 + e^2 \wedge e^4, \quad de^8 = e^1 \wedge e^4 - e^2 \wedge e^3.$$

Define an invariant complex structure by $Ie^{2i-1} = e^{2i}$, for $i = 1, \dots, 4$. We warn the reader that this complex structure is different from the natural extension of the bi-invariant complex structure on the complex Heisenberg group.

Consider the invariant 2-forms

$$\begin{aligned} \Theta &= e^1 \wedge e^8 - e^2 \wedge e^7 + e^3 \wedge e^6 - e^4 \wedge e^5, \\ \Omega_2 &= e^1 \wedge e^5 - e^2 \wedge e^6 + e^3 \wedge e^7 - e^4 \wedge e^8, \\ \Omega_3 &= e^1 \wedge e^6 + e^2 \wedge e^5 + e^3 \wedge e^8 + e^4 \wedge e^7. \end{aligned}$$

Define a metric g_Θ with Θ as its fundamental form and linear endomorphisms S and T by

$$\Omega_2(X, Y) = \Theta(SX, IY), \quad \Omega_3(X, Y) = \Theta(TX, IY).$$

Then the metric connection for g_Θ preserves (I, S, T) and hence we have a (flat) hypersymplectic structure.

Now the 2-form Ω given by (6) is again a special 2-form. It amounts to the following choice:

$$\begin{aligned} &E^i = e^i, \quad i = 1, \dots, 4, \\ &E^5 = -e^8 - \tfrac{1}{2}h_1 e^2 + h_4 e^3 - h_3 e^4, \quad E^6 = e^7 + \tfrac{1}{2}h_1 e^1 + h_3 e^3 + h_4 e^4, \\ &E^7 = -e^6 - \tfrac{1}{2}h_2 e^4, \quad E^8 = e^5 + \tfrac{1}{2}h_2 e^3. \end{aligned}$$

By Theorem 2.2, the neutral metric

$$g = E^1 \cdot E^6 - E^2 \cdot E^5 + E^3 \cdot E^8 - E^4 \cdot E^7, \tag{8}$$

is Ricci-flat. By Corollary 2.3, for any solution $h_i(x_1, x_2, x_3, x_4)$ of (7), (I, g, Ω) is a neutral Calabi-Yau structure.

Next, we define linear endomorphisms J and K by

$$\Omega_2(X, Y) = \Omega(JX, IY), \quad \text{and} \quad \Omega_3(X, Y) = \Omega(KX, IY).$$

In order for (I, J, K) to form a hypersymplectic structure, the functions h_i are subject to additional constraints. For instance, if $h_1 = h_2$ is independent of x_3, x_4 and if h_3 and h_4 are constants, then these constraints are satisfied.

Example 2. The structural equations for the second group G_2 are

$$de^i = 0, \quad i = 1, \dots, 5, \text{ and}$$
$$de^6 = e^1 \wedge e^2 + e^3 \wedge e^4, \quad de^7 = e^1 \wedge e^3, \quad de^8 = e^1 \wedge e^4.$$

Define a complex structure by $Ie^{2i-1} = e^{2i}$, for $i = 1, \dots, 4$. The forms

$$\Theta = -e^1 \wedge e^8 + e^2 \wedge e^7 + e^3 \wedge e^6 - e^4 \wedge e^5,$$
$$\Omega_2 = -e^1 \wedge e^7 + e^2 \wedge e^8 - e^3 \wedge e^5 + e^4 \wedge e^6,$$
$$\Omega_3 = e^1 \wedge e^8 - e^2 \wedge e^7 - e^3 \wedge e^6 + e^4 \wedge e^5,$$

are invariant, closed and non-degenerate. They form a hypersymplectic structure with a flat metric. Again, we use (6) to find a new special form Ω. With the complex structure I, we get a new metric. In terms of Proposition 2.1, it amounts to

$$E^i = e^i, \quad i = 1, \dots, 4,$$
$$E^5 = e^8 - \tfrac{1}{2}h_1 e^2 + h_4 e^3 - h_3 e^4, \quad E^6 = -e^7 + \tfrac{1}{2}h_1 e^1 + h_3 e^3 + h_4 e^4,$$
$$E^7 = -e^6 - \tfrac{1}{2}h_2 e^4, \quad E^8 = e^5 + \tfrac{1}{2}h_2 e^3.$$

As in the last example, when $h_1 = h_1(x_1, x_2)$, $h_2 = h_2(x_3, x_4)$ and h_3, h_4, we find a hypersymplectic structure I, J, K so that the fundamental 2-forms are $\Omega, \Omega_2, \Omega_3$.

Example 3. The next two examples are central extensions of 2-step nilpotent Lie groups of dimension 6. Let G_3 be the 2-step nilpotent Lie group $H_3 \times H_3 \times \mathbb{R}^2$. Its structure equations are

$$de^i = 0, \quad i = 1, \dots, 6, \qquad de^7 = e^1 \wedge e^2, \qquad de^8 = e^3 \wedge e^4.$$

Define an invariant complex structure by $Ie^1 = e^2$, $Ie^3 = e^4$, $Ie^5 = e^7$, $Ie^6 = e^8$. A process similar to the last example using

$$\Theta = e^1 \wedge e^5 + e^2 \wedge e^7 + e^3 \wedge e^6 + e^4 \wedge e^8,$$
$$\Omega_2 = -e^1 \wedge e^7 - e^2 \wedge e^5 - e^3 \wedge e^8 - e^4 \wedge e^6,$$
$$\Omega_3 = -e^1 \wedge e^5 - e^2 \wedge e^7 - e^3 \wedge e^6 - e^4 \wedge e^8$$

shows that if we use,

$$E^i = e^i, \quad i = 1, 2, 3,$$
$$E^4 = -e^4, \quad E^5 = -e^5 - \tfrac{1}{2}h_1 e^2 + h_4 e^3 - h_3 e^4,$$
$$E^6 = -e^7 + \tfrac{1}{2}h_1 e^1 + h_3 e^3 + h_4 e^4,$$
$$E^7 = -e^6 - \tfrac{1}{2}h_2 e^4, \quad E^8 = -e^8 + \tfrac{1}{2}h_2 e^3$$

on $\Gamma_3 \backslash G_3$ as in Eq. (8) and if h_i satisfies (7), we get a non-flat neutral Calabi-Yau metric. For example, if $h_4 = 0$, we consider

$$Je^1 = e^1, \quad Je^2 = -e^2, \quad Je^3 = e^3, \quad Je^4 = -e^4,$$
$$Je^5 = e^5 + h_1 e^2 + h_3 e^4, \quad Je^6 = e^6 + h_2 e^4 + h_3 e^2,$$
$$Je^7 = -e^7 + h_1 e^1 + h_3 e^3, \quad Je^8 = -e^8 + h_2 e^3 + h_3 e^1.$$

The structures I and J together with $K = IJ$ give a hypersymplectic structure on $\Gamma_3 \backslash G_3$.

Example 4. The structure equation for our last example is

$$de^i = 0, \quad i = 1, \ldots, 6, \qquad de^7 = e^1 \wedge e^2, \qquad de^8 = e^1 \wedge e^4 + e^2 \wedge e^3.$$

A complex structure is defined by $Ie^1 = e^2, Ie^3 = -e^4, Ie^5 = e^7, Ie^6 = e^8$. The neutral metric determined by (8) and

$$\begin{aligned} &E^i = e^i, \quad i = 1, \ldots, 4, \quad E^5 = -e^8 - \tfrac{1}{2}h_1 e^2 - h_4 e^3 + h_3 e^4, \\ &E^6 = e^6 + \tfrac{1}{2}h_1 e^1 + h_3 e^3 + h_4 e^4, \quad E^7 = e^7 + \tfrac{1}{2}h_2 e^4, E^8 = -e^5 + \tfrac{1}{2}h_2 e^3 \end{aligned}$$

is non-flat neutral Calabi-Yau if h_i satisfies (7). For example, if $h_i = 0, i = 2, 3, 4$, using

$$\begin{aligned} \Theta &= e^1 \wedge e^8 - e^2 \wedge e^6 - e^3 \wedge e^7 - e^4 \wedge e^5, \\ \Omega_2 &= e^1 \wedge e^6 - e^2 \wedge e^8 - e^3 \wedge e^5 - e^4 \wedge e^7, \\ \Omega_3 &= e^1 \wedge e^8 + e^2 \wedge e^6 - e^3 \wedge e^7 + e^4 \wedge e^5, \end{aligned}$$

we define a real structure J by

$$\begin{aligned} &Je^1 = e^1, Je^2 = -e^2, Je^3 = e^3, Je^4 = -e^4, \\ &Je^5 = -e^5, Je^7 = e^7, \qquad Je^6 = -e^6 - h_1 e^1, \qquad Je^8 = e^8 + h_1 e^2. \end{aligned}$$

The structures I, J together with $K = IJ$ yield a hypersymplectic structure on $\Gamma_4 \backslash G_4$.

4. Cotangent Bundles of Nilpotent Lie Groups

Since not all Kodaira manifolds admit a closed non-degenerate (1, 1)-form, we shall present a method to construct a Kodaira manifold of dimension $8n$ with an invariant, closed and non-degenerate (1, 1)-form starting from a Kodaira manifold of dimension $4n$.

Let G be a simply connected real Lie group of dimension m and let T^*G be the cotangent bundle of G. It is possible to identify G with the zero section in T^*G and $\mathfrak{g}^*$ with the fibre over a neutral element $(0, e)$ of T^*G. Therefore, we will identify $T_{0,e}(T^*G)$ with $\mathfrak{g}^* \times \mathfrak{g}$.

Let ω be the skew-symmetric bilinear form defined on $\mathfrak{g}^* \times \mathfrak{g}$ by setting

$$\omega((\theta, X), (\theta', X')) = \theta(X') - \theta'(X), \tag{9}$$

for any $\theta, \theta' \in \mathfrak{g}^*$ and for any $X, X' \in \mathfrak{g}$. If one considers the standard structure of the Lie group on T^*G with composition law given by

$$(\theta_1, \gamma_1) \circ (\theta_2, \gamma_2) = (L^*_{\gamma_1^{-1}}\theta_2 + R^*_{\gamma_2}\theta_1, \gamma_1\gamma_2), \quad \theta_i \in \mathfrak{g}, \quad \gamma_i \in G, \quad i = 1, 2,$$

(where L_γ and R_γ denote the left and right translation by γ, respectively), then one can check that ω is closed only if G is abelian.

By [3, 4] it is possible to deform the Lie group structure of T^*G in such a way that ω becomes closed, giving a symplectic structure on T^*G.

More precisely, suppose that G has an affine structure, i.e. it has a flat torsion free linear connection ∇ which is invariant under left translations. Using the above affine structure and identifying T^*G with $\mathfrak{g}^* \times G$, Boyom endowed T^*G with a structure of Lie group such that $d\omega = 0$, [3]. Indeed, he considered on the Lie algebra $\mathfrak{h} = \mathfrak{g}^* \times \mathfrak{g}$ of T^*G the following bracket operation:

$$[(\theta, X), (\theta', X')] = (\theta \circ \nabla_{X'} - \theta' \circ \nabla_X, [X, X']),$$

for any $\theta, \theta' \in \mathfrak{g}^*$ and for any $X, X' \in \mathfrak{g}$. Using the fact that the torsion of ∇ is zero, one can easily check that the non-degenerate left-invariant 2-form defined by ω is closed and it becomes a left-invariant symplectic structure on T^*G.

Let $\{X_1, \dots, X_m\}$ be a basis of $\mathfrak{g}$ and $\{\theta^1, \dots, \theta^m\}$ its dual. Consider the basis $\{e_i = (\theta^i, 0), e_{m+i} = (0, X_i), i = 1, \dots, m\}$ of $\mathfrak{h}$ and let $\{e^1, \dots, e^{2m}\}$ be the dual basis. Then $\omega = \sum_{i=1}^m e^i \wedge e^{m+i}$.

If in addition, $m = 2n$ and I is an invariant complex structure on G such that $\nabla I = 0$, we define $\mathbf{I}$ by

$$\mathbf{I}(\theta, X) = (I\theta, IX), \qquad \text{for } (\theta, X) \in \mathfrak{g}^* \times \mathfrak{g}. \tag{10}$$

This is a complex structure on T^*G invariant with respect to the above Lie group structure. With respect to this complex structure, ω is of type (1,1) [12]. In the case of 2-step nilpotent Lie groups we have the following

Proposition 4.1. *Let* $(M = \Gamma\backslash G, I)$ *be a Kodaira manifold of dimension* $4n$*. If* ∇ *is a* G*-invariant torsion-free connection such that:*

(i) $\nabla I = 0$
(ii) $\nabla X = 0$, *for all* $X \in \mathfrak{z}$
(iii) $\mathfrak{z} = span\{\nabla_X Y : X, Y \in \mathfrak{g}\}$,

then there exists on T^*G *an invariant symplectic form* ω *of type* $(1, 1)$ *with respect to the induced complex structure* $\mathbf{I}$.

Any compact quotient of T^*G *is a Kodaira manifold of dimension* $8n$ *and has a neutral Calabi-Yau structure defined by* $\omega + \pi^*\beta$*, where* β *is a closed* $(1, 1)$*-form on the base torus of dimension* $4n$ *with special Lagrangian torias fibers.*

Proof. As in the proof of Proposition 2.1, it is possible to choose a basis of left-invariant 1-forms $\{e^1, \dots, e^{4n}\}$ on G which satisfies the conditions (a), (b) and (c) and such that the center $\mathfrak{z}$ of $\mathfrak{g}$ coincides with span$\langle e_{2n+1}, \dots, e_{4n}\rangle$. Due to (ii) and (iii), ∇ is flat. The previous construction equips T^*G with a Lie group structure. Since G is a 2-step nilpotent Lie group, one deduces that T^*G is also a 2-step nilpotent. Let $\{e^1, \dots, e^{4n}\}$ be the dual basis. Then, we can consider

$$f_i = (e^i, 0), \qquad f_{4n+j} = (0, e_j),\ i, j = 1, \dots, 4n$$

as a basis of the Lie algebra $\mathfrak{h}$ of T^*G. Using (9) and (10), we find that ω is a symplectic form of type (1,1) with respect to $\mathbf{I}$.

Moreover, (θ, X) belongs to the center of $\mathfrak{h}$ if and only if $X \in \mathfrak{z}$ and $\theta \circ \nabla_Y = 0$, for any $Y \in \mathfrak{g}$. By (iii), the center of $\mathfrak{h}$ coincides with span$\langle f_1, \dots, f_{2n}, f_{6n+1}, \dots, f_{8n}\rangle$. It is now evident that ω vanishes on the center of $\mathfrak{h}$ and $\mathbf{I}$ preserves the center. Thus, any compact quotient of T^*G is a Kodaira manifold with center of dimension $4n$.

If β is a closed $(1, 1)$-form on the base torus (not necessarily T^*G-invariant), one can check that $\Omega = \omega + \pi^*\beta$ is a closed special 2-form. By Theorem 2.2, it defines a neutral Calabi-Yau structure on the compact quotient of T^*G. □

5. The Cotangent Bundle of $H_3 \times \mathbb{R}$

The last proposition can be applied to compact quotients of T^*G, where G is either $H_3 \times \mathbb{R}$ or an 8-dimensional nilpotent Lie group endowed with an invariant hypercomplex structure (except the one with 5-dimensional center). Indeed, by [9] the Obata connection associated to any invariant hypercomplex structure $\{I_1, I_2\}$ on such groups satisfies the conditions (i),(ii),(iii) with I either of the two complex structures. We focus on $H_3 \times \mathbb{R}$.

Let $\mathfrak{h}_3$ be the Lie algebra of the real 3-dimensional Heisenberg group H_3. The 4-dimensional 2-step nilpotent Lie algebra $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathbb{R}$ has a basis $\{X, Y, Z, W\}$ such that

$$[X, Y] = W.$$

The complex structure I, defined by $IX = Y, IZ = W$, preserves the center $\mathfrak{z} = \operatorname{span}\langle Z, W\rangle$.

Any left-invariant torsion-free flat connection such that the conditions (i),(ii),(iii) of Proposition 4.1 hold is given by

$$\begin{aligned} &\nabla^{a,b}_X X = aZ + bW, \qquad \nabla^{a,b}_Y X = -bZ + (a-1)W, \\ &\nabla^{a,b}_X Y = -bZ + aW, \qquad \nabla^{a,b}_Y Y = (1-a)Z - bW, \qquad a, b \in \mathbb{R}. \end{aligned}$$

According to the construction in the last section, these connections define Lie group structures $G_{a,b}$ on the manifold $T^*(H_3 \times \mathbb{R})$. The complex structure I on G induces a $G_{a,b}$-invariant complex structure $\mathbf{I}$ on $T^*(H_3 \times \mathbb{R})$.

Considering the standard basis of the Lie algebra $\mathfrak{g}_{a,b}$ of $G_{a,b}$,

$$\begin{aligned} &e_1 = (X^*, 0), e_2 = (Y^*, 0), e_3 = (Z^*, 0), e_4 = (W^*, 0), \\ &e_5 = (0, X), e_6 = (0, Y), e_7 = (0, Z), e_8 = (0, W), \end{aligned}$$

we find for the dual basis $\{e^i, i = 1, \dots, 8\}$ of left-invariant 1-forms the structure equations

$$\begin{aligned} de^1 &= -ae^3 \wedge e^5 + be^3 \wedge e^6 - be^4 \wedge e^5 + (1-a)e^4 \wedge e^6, \\ de^2 &= be^3 \wedge e^5 + (a-1)e^3 \wedge e^6 - ae^4 \wedge e^5 + be^4 \wedge e^6, \\ de^8 &= -e^5 \wedge e^6, \qquad de^i = 0 \quad i = 3, \dots, 7. \end{aligned}$$

The Lie algebras $\mathfrak{g}_{a,b}$ depend only on whether $a^2 + b^2 - a$ vanishes or not. In both cases the corresponding Lie algebra belongs to the list of 8-dimensional nilpotent Lie algebras which admit invariant hypercomplex structures [8]. Then, for any $a, b \in \mathbb{R}$, there exists a discrete subgroup $\Gamma_{a,b}$ such that $\Gamma_{a,b}\backslash G_{a,b}$ is a Kodaira manifold.

The induced complex structure $\mathbf{I}$ on $G_{a,b}$, given by $\mathbf{I}\, e^{2i-1} = e^{2i}, i = 1, \dots, 4$, preserves the center $\mathfrak{z} = \operatorname{span}\langle e_1, e_2, e_7, e_8\rangle$.

According to Proposition 4.1 we now have a special 2-form

$$\omega = e^1 \wedge e^5 + e^2 \wedge e^6 + e^3 \wedge e^7 + e^4 \wedge e^8.$$

If one considers the 2-form Ω given by (6) with $\Theta = \omega$ and

$$E^1 = e^3, E^2 = e^4, E^3 = e^5, E^4 = e^6$$

then the metric g associated to Ω on $\Gamma_{a,b}\backslash G_{a,b}$ can be written as

$$g = E^1 \cdot E^6 - E^2 \cdot E^5 + E^3 \cdot E^8 - E^4 \cdot E^7,$$

with

$$\begin{aligned} &E^5 = -\tfrac{h_1}{2} e^4 + h_4 e^5 - h_3 e^6 - e^7, \qquad E^6 = \tfrac{h_1}{2} e^3 + h_3 e^5 + h_4 e^6 - e^8, \\ &E^7 = e^1 - \tfrac{h_2}{2} e^6, \qquad E^8 = e^2 + \tfrac{h_2}{2} e^5. \end{aligned}$$

The forms $\{E^i, i = 1, \dots, 8\}$ satisfy the conditions (1). Then, for any solution h_i, $i = 1, 2, 3, 4$, of (7), g is a neutral Calabi-Yau metric. Moreover, the 2-forms

$$\begin{aligned}\Omega_2 &= -e^1 \wedge e^4 - e^2 \wedge e^3 - e^5 \wedge e^8 - e^6 \wedge e^7,\\ \Omega_3 &= -e^2 \wedge e^4 + e^1 \wedge e^3 + e^5 \wedge e^7 - e^6 \wedge e^8,\end{aligned} \tag{11}$$

are closed and they define a hypersymplectic structure on $\Gamma_{a,b}\backslash G_{a,b}$. For example, if $h_1 = h_2$, Ω_2 together with the metric g, defines a real structure $\mathbf{J}$. In terms of 1-forms, $\mathbf{J}$ is given by

$$\begin{aligned}&\mathbf{J}\,e^1 = h_1e^4 - h_4e^5 + h_3e^6 + e^7, \qquad \mathbf{J}\,e^2 = h_1e^3 + h_3e^5 + h_4e^6 - e^8,\\ &\mathbf{J}\,e^3 = -e^5, \qquad \mathbf{J}\,e^4 = e^6, \qquad \mathbf{J}\,e^5 = -e^3, \qquad \mathbf{J}\,e^6 = e^4,\\ &\mathbf{J}\,e^7 = e^1 - h_4e^3 - h_3e^4 - h_2e^6, \qquad \mathbf{J}\,e^8 = -e^2 - h_3e^3 + h_4e^4 - h_2e^5.\end{aligned}$$

Together with $\mathbf{I}$ and $\mathbf{K} = \mathbf{I}\,\mathbf{J}$, it gives a hypersymplectic structure on $\Gamma_{a,b}\backslash G_{a,b}$.

It is interesting to note that the construction in Proposition 4.1 allows more freedom. In the proof of Proposition 4.1 we used that if the dual basis $\{f^j, j = 1, \dots, 8\}$ for $G_{a,b}$ satisfies

- $\mathbf{I}\,f^{2i-1} = f^{2i}, i = 1, \dots, 4$
- $\mathfrak{z}(G_{a,b}) = \mathrm{span}\langle f_1, f_2, f_7, f_8\rangle$

then we can write $\omega = \sum_{j=1}^4 f^j \wedge f^{4+j}$. With this in mind and setting $f^i = e^i$, $i = 1, 2, 7, 8$, we find through extensive calculations that the most general form of ω depends on the Lie group structures of $G_{a,b}$. Let

$$\begin{aligned}\omega = e^1 &\wedge (g_3e^3 + g_4e^4 + g_5e^5 + g_6e^6 + g_7e^7 + g_8e^8)\\ &+e^2 \wedge (-g_4e^3 + g_3e^4 - g_6e^5 + g_5e^6 - g_8e^7 + g_7e^8)\\ &+(-g_7e^1 + g_8e^2 + g_5e^3 - g_6e^4 + g_9e^5 + g_{10}e^6) \wedge e^7\\ &+(-g_8e^1 - g_7e^2 + g_6e^3 + g_5e^4 - g_{10}e^5 + g_9e^6) \wedge e^8\end{aligned}$$

with $g_i \in \mathbb{R}$. Now we describe the conditions for ω to be a special 2-form.

In order for $\omega = \omega_{a,b}$ to be closed we should have

$$-bg_3 + ag_4 = 0 \qquad (1-a)g_3 - bg_4 = 0.$$

These constraints give four cases:

- $a^2 + b^2 - a \neq 0 \Rightarrow g_3 = g_4 = 0$,
- $a^2 + b^2 - a = 0, b \neq 0 \Rightarrow g_3 = \frac{a}{b}g_4$,
- $a^2 + b^2 - a = 0, b = 0, a = 0 \Rightarrow g_3 = 0$,
- $a^2 + b^2 - a = 0, b = 0, a = 1 \Rightarrow g_4 = 0$.

In order for $\omega_{a,b}$ to be non-degenerate the following extra conditions must hold for each of the above cases:

- $(g_5^2 + g_6^2)^2 \neq 0$,
- $g_4^2(\frac{a^2}{b^2} + 1)(g_9^2 + g_{10}^2) + 2g_4(g_{10} - \frac{a}{b}g_9)(g_5^2 + g_6^2) + (g_5^2 + g_6^2)^2 \neq 0$,
- $g_4^2(g_9^2 + g_{10}^2) + 2g_4g_{10}(g_5^2 + g_6^2) + (g_5^2 + g_6^2)^2 \neq 0$,
- $g_3^2(g_9^2 + g_{10}^2) - 2g_3g_9(g_5^2 + g_6^2) + (g_5^2 + g_6^2)^2 \neq 0$.

Thus, we have proven:

Theorem 5.1. *To any solution of* (7) *for fixed* a, b *there exist:*

- *a* 6*-parameter family of neutral Calabi-Yau structures if* $a^2 + b^2 - a \neq 0$,
- *a* 7*-parameter family of neutral Calabi-Yau structures if* $a^2 + b^2 - a = 0$.

If we consider Ω_2 as in (11), then Ω_2 defines a real structure $\mathbf{J}$:

$$\begin{aligned}
\mathbf{J}\, e^2 &= g_4 e^1 + g_3 e^2 + h_1 e^3 + h_3 e^5 + h_4 e^6 + g_6 e^7 - g_5 e^8,\\
\mathbf{J}\, e^1 &= -g_3 e^1 + g_4 e^2 + h_1 e^4 - h_4 e^5 + h_3 e^6 + g_5 e^7 + g_6 e^8,\\
\mathbf{J}\, e^3 &= -g_3 e^3 - g_4 e^4 - g_5 e^5 - g_6 e^6,\\
\mathbf{J}\, e^4 &= -g_4 e^3 + g_3 e^4 - g_6 e^5 + g_5 e^6,\\
\mathbf{J}\, e^5 &= -g_5 e^3 + g_6 e^4 - g_9 e^5 - g_{10} e^6,\\
\mathbf{J}\, e^6 &= g_6 e^3 + g_5 e^4 - g_{10} e^5 + g_9 e^6,\\
\mathbf{J}\, e^7 &= g_5 e^1 - g_6 e^2 - h_4 e^3 - h_3 e^4 - h_2 e^6 - g_9 e^7 + g_{10} e^8,\\
\mathbf{J}\, e^8 &= -g_6 e^1 - g_5 e^2 - h_3 e^3 + h_4 e^4 - h_2 e^5 + g_{10} e^7 + g_9 e^8.
\end{aligned}$$

Given a solution of (7) with $h_1 = h_2$ and such that $a^2 + b^2 - a \neq 0$, then apriori the resulting hypersymplectic structure depends on a and b, but in fact there are only two different such hypersymplectic structures.

The case $a^2 + b^2 - a = 0$ leads to more complicated conditions for $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K} = \mathbf{I}\,\mathbf{J}$ to be well defined. We can show that these conditions are satisfied by a 1-parameter family of hypersymplectic structures $(\Omega_1, \Omega_2, \Omega_3)$ for any solution of (7) with

$$\begin{aligned}
\Omega_1 = {} & e^1 \wedge (\cos\theta\, e^3 + \sin\theta\, e^5) + e^2 \wedge (\cos\theta\, e^4 + \sin\theta\, e^6)\\
& + (\sin\theta\, e^3 - \cos\theta\, e^5) \wedge e^7 + (\sin\theta\, e^4 - \cos\theta\, e^6) \wedge e^8, \quad \theta \in \mathbb{R}
\end{aligned}$$

and Ω_2, Ω_3 defined by (11).

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