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SELF-DUALITY, TWISTOR THEORY, ITS GENERALIZATION AND APPLICATION

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§1 Twistor theory has its root in mathematical physics [P]. In 1978, Atiyah et. al. [A] developed this theory on 4-dimensional Riemannian self-dual manifolds. Then Salamon realized that quaternionic structure on 4n dimensional manifold can be considered as a high dimensional analogue of the (anti-)self-dual conformal structure on a 4-fold [S_1]. We shall discuss some results related to the author's work in this area.

A four dimensional oriented Riemannian manifold is self-dual [A] if the Weyl tensor of the Levi-Civita connection is invariant under the action of the Hodge $*$-operator. As the $*$-operator on 2-forms over 4-fold and the Weyl tensor are conformal invariants, a conformal change of self-dual metric is again self-dual. Fundamental nontrivial examples are the Euclidean 4-sphere, the complex projective plane with Fubini-Study metric.

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More exotic example is the K3 surface with Calabi-Yau metric and orientation opposite to the canonical one [B].

Twistor space is a complex manifold associated to a self-dual manifold naturally as follow: the holonomy of an oriented 4-dimensional Riemannian manifold, X, is contained in the group SO(4), i.e. $SU(2)xSU(2)/\mathbb{Z}_2$. Each copy of SU(2) has an irreducible representation of dimension 3. Associated to the principal bundle of frame on X are two rank 3 bundles. They are the bundles of self-dual and anti-self-dual 2-forms respectively. The total space, Z, of the unit sphere bundle of anti-self-dual 2-forms is the twistor space. It can also be considered as the associated fibre bundle induced by the action of SU(2) acting on the projective space of the 2-dimensional representation of SU(2). As one can use the metric to identify 2-forms with skew-symmetric endomorphism on tangent bundle, points on twistor space are almost complex structures on X. Using the induced connection of the twistor space to split the tangent space into horizontal and vertical parts, one can choose the complex structure at a point J on the twistor space to be the standard Riemann sphere structure on the vertical part and the J on the horizontal part. On a self-dual manifold, the obstruction to the integrability vanishes, and therefore Z is a complex manifold [A].

Moreover, the fibres of the projection, π, from Z onto X are holomorphic rational curves. This fibration is called the twistor fibration and the fibres are called the twistor lines.

Since the Hodge $*$-operator on 2-forms over a four dimensional manifold is conformally invariant, the holomorphic geometry depends only on the conformal geometry on X. On the other hand, given a twistor space, i.e. a complex manifold foliated by appropriate family of rational curves, one can always find a self-dual conformal class on a 4-fold [B]. This one-to-one correspondence is usually called the twistor correspondence or Penrose correspondence.

We shall discuss how the twistor theory and its generalization can be applied to construct or classify self-dual manifolds and its higher dimensional analogue.

§2 As the twistor correspondence is one-to-one, one may try to construct self-dual manifolds by producing a twistor space. For example, $\mathbb{CP}^3$ is the twistor space over S^4, the projection is simply the Hopf fibration. On the complex projective plane $\mathbb{CP}^2$, the twistor space is the flag F(1,2) of lines and planes in $\mathbb{C}^3$. The projection from F(1,2) onto $\mathbb{CP}^2$ is to take the intersection of a given plane and the orthogonal complement of a given line in the given plane. With this approach, new examples of self-dual structures were found on the connected-sum of projective planes [DF, P_1]. It was shown in [P_1] that a small resolution of the intersection of two quadrics with four nodes in $\mathbb{CP}^5$ is the twistor spaces associated to the connected-sum of two projective planes. The fibres of the projection from the small resolution to the connected-sum are conics. Algebraically, the intersection is given by:

$$Q_0 = \{z \in \mathbb{CP}^5 : \alpha z_0^2 + \alpha z_1^2 + \beta z_2^2 + \gamma z_3^2 + \lambda z_4^2 + \lambda z_5^2 = 0\};$$

and $Q_\infty = \{z \in \mathbb{CP}^5 : z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0\}$,

where $\alpha > \beta > \gamma > \lambda$. There are two interesting limits in this family of intersection of quadrics described by Donaldson and Kronheimer [D,K] respectively as follows:

1) when $\beta = \gamma$, and α approaches to λ by passing through the point at infinity, the above intersection is deformed to the intersection of

$$Q_0 = \{z \in \mathbb{CP}^5 : z_2^2 + z_3^2 = 0\};$$

and $Q_\infty = \{z \in \mathbb{CP}^5 : z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0\}$.

This intersection can be considered as gluing the two quadrics

$$F_1 = \{z \in \mathbb{CP}^5 : z_0^2 + z_1^2 + z_4^2 + z_5^2 = 0,\ z_2 - iz_3 = 0\}$$

and $$F_2 = \{z \in \mathbb{CP}^5 : z_0^2 + z_1^2 + z_4^2 + z_5^2 = 0,\ z_2 + iz_3 = 0\}$$

along the quadric surfaces

$$Q = \{z \in \mathbb{CP}^5 : z_0^2 + z_1^2 + z_4^2 + z_5^2 = 0,\ z_2 = 0,\ z_3 = 0\}.$$

Note that F_1 and F_2 can be considered as the blowing-up of the twistor space F(1,2) along a twistor line whose exceptional divisor is exactly the quadric surface Q.

In general, Donaldson and Friedman [DF] recently found a general construction of self-dual structure on the connected-sum of two self-dual manifolds. In short, they blow up a twistor line from the twistor space of each manifold. The exceptional divisors are quadric surface because the normal bundle of a twistor line is $H \oplus H$, where **H** is the hyperplane bundle on a complex projective curve. When there is no obstruction, the complex space obtained by gluing the two twistor spaces along the quadric surfaces can be deformed to a twistor space. Then the Penrose correspondence provides the last step to produce a self-dual structure. Examples on which their construction can be carried out are the connected-sums of the complex projective planes.

2) when $\alpha=\beta$, and $\gamma=\lambda$, the intersection is deformed to be

$$Q_0 = \{z \in \mathbb{CP}^5 : z_0^2 + z_1^2 + z_2^2 = 0\}$$

and $Q_\infty = \{z \in \mathbb{CP}^5 : z_3^2 + z_4^2 + z_5^2 = 0\}$.

This is the image of $\mathbb{CP}^3$ in $\mathbb{CP}^5$ via the map

$$[z_0 : z_1 : z_2 : z_3] \to [z_0^2 : z_0 z_1 : z_1^2 : z_2^2 : z_2 z_3 : z_3^2].$$

i.e. the singular space $\mathbb{CP}^3/\mathbb{Z}_2$. Although this space is not smooth, it is the twistor space of the orbifold $S^4/\mathbb{Z}_2$. This orbifold is not only self-dual but also Einstein. But this orbifold can be considered as trivial in the sense that it is the quotient of a smooth manifold by finite group of automorphism.

On the other hand, Galicki and Lawson found a family of nontrivial self-dual Einstein orbifold [GL]. From a generalization of the symplectic reduction and hyper-Kähler reduction [HK], they developed the so-called quaternionic reduction on quaternionic Kähler manifolds and found new examples of compact self-dual Einstein orbifolds and quaternionic Kähler orbifolds with positive scalar curvature. By definition, a quaternionic Kähler manifold is a Riemannian manifold whose holonomy is in the group $Sp(n)Sp(1)$. It is known for some time that quaternionic Kähler manifolds are Einstein space [B]. However, Salamon [S_1,S_2] found that quaternionic Kähler manifold can be considered as a high dimensional generalization of (anti-)self-dual Einstein manifold in the sense that the associated fibre bundle of the action of Sp(1) on $\mathbb{CP}^1$ is a manifold with a natural integrable complex structure. As in the four dimensional case, the total space of this fibre bundle is called the twistor space associated to the quaternionic Kähler manifold. Moreover, the Levi-Civita connection induces a horizontal distribution on the twistor space. When the scalar curvature is not zero, this distribution is the kernel of a complex contact structure [S_2]. Basic example of quaternionic Kähler manifold is projective quaternionic $\mathbb{HP}^n$. Its twistor space is $\mathbb{CP}^{2n+1}$. He further generalized to define a quaternionic manifold as a 4n-dimensional manifold, $n \geq 2$, with a $GL(n,\mathbb{H})Sp(1)$-structure admitting a torsion-free connection. The definition is tailored so that the associated twistor space has a natural integrable complex structure. This type of geometry should be considered as a high dimensional version of a (anti-)self-dual conformal structure on a 4-fold. To complete the Penrose correspondence, the author and Pedersen [PP_2] recently proved that a twistor space determines a quaternionic structure on a manifold.

This observation enables us to find new examples of quaternionic manifolds. We can also mimic the four dimensional case and prove that a twistor space with appropriate contact structure will generate a quaternionic Kähler manifold, a large family of new examples construced by this twistorial method is found by LeBrun [L_1] who also independently proved this result. As the constructions of LeBrun and Galicki and Lawon are already discussed in another survey article [S_4], we skip it here.

§3 Since the holomorphic data on a twistor space determines a quaternionic structure, it can not only help to construct a self-dual or quaternionic structure but also provide a way to classification. The first such theorem is Hitchin's result [H_1] that a compact Kählerian twistor space on a self-dual manifold is necessarily the twistor space of the Euclidean sphere S^4 and the twistor space of the projective plane with the Fubini-Study metric. As the twistor space of a self-dual Einstein manifold with positive scalar curvature is Kählerian, this theorem implies that the Riemannian manifolds are the only examples of compact self-dual Einstein manifolds with positive scalar curvature, a result also proved by Friedrich and Kurke [FK]. The outline of Hitchin's method is the following:

on a twistor space, there is always an anti-holomorphic involution σ induced by antipodal map on the twistor lines. It is usually called a real structure. If $[\omega]$ is a Kähler class, then $[\omega]-\sigma^*[\omega]$ is a σ-invariant class represented by positive form. As σ-invariant class is generated by the first Chern class of the twistor space, the anticanonical bundle is positive. In particular, the Chern number c_1^3 is positive. With the Hodge symmetry, it imposed topological constraints on the twistor space and hence on the self-dual manifold. To be precise, the signature of X must be 0, 1, 2, or 3. Moreover, from the tautalogical construction of the twistor space, one can find a holomomrphic line bundle $\mathbf{K}^{-\frac{1}{2}}$ such that $(\mathbf{K}^{-\frac{1}{2}})^2=\mathbf{K}^{-1}$. This is also a positive line bundle. With the Riemann-Roch formula and the given topological constraints, one can prove that the Euler number of the bundle $\mathbf{K}^{-\frac{1}{2}}$ is positive. Then an application of the vanishing theorem of Kodaira will implies that this bundle has global nontrivial sections. The main body of the work is to show that the associated map of this line bundle is an embedding. Finally, when the signature is equal to 2 and 3, he identified the image of the embedding as the complete intersection of two smooth quadrics in $\mathbb{CP}^5$ and the double covering of $\mathbb{CP}^3$ branched along a smooth quartic surface. However, they cannot be a twistor space because the topological data are not compatible. When the signature is equal to 0 or 1, he identified the twistor space as stated by computing the group of conformal transformation of the self-dual manifold. This computation is possible because the twistor correspondence identifies this group as the real form of the group of holomorphic transformation on the twistor space.

This entire programme is adapted by the author and Salamon [PS] to show that any 8-dimensional compact quaternionic Kähler manifold with positive scalar curvature is symmetry. In fact, with the assumption that the scalar curvature is positive, one can construct a canonical Einstein Kähler metric with positive scalar curvature on the twistor space. Then a natural third root of the anticanonical bundle, $K^{-1/3}$, is positive. Except when the twistor space is the complex projective space, this bundle is the generator of the Picard group on the twistor space [S_2]. In other words, the twistor space is a 5-dimensional Fano-manifold with index 3. In this case, it is not hard to apply the Riemann-Roch formula and the Serre duality to deduce that the bundle $K^{-1/3}$ has global sections. The work again is to show that the associated map of this bundle is an embedding. On the other hand, the algebra of Killing vector field on X is the real form of the space of sections of $K^{-1/3}$. When this space is sufficiently large, then the manifold is homogeneous and hence symmetric [Al]. When it is not large, we are able to identify the image of the embedding and hence determine the twistor spaces of all 8-dimensional quaternionic Kähler manifolds with positive scalar curvature.

It remains to see whether there are non-symmetric compact quaternionic Kähler manifolds with positive scalar curvature in higher dimension. However, LeBrun [L_2] shows that any compact quaternionic Kähler manifold with positive scalar curvature is rigid in the sense that such geometrical structure has no deformation on the given differentiable manifold. This is the consequence of his result that there is no contact deformation on the twistor space when the scalar curvature on X is positive.

The idea of Hitchin can also be further developed in the four dimensional case [P_3, P_4, V]. In fact, when a compact twistor space is Moishezon, the self-dual conformal class must contain a metric with constant positive scalar curvature. The basic reason is that the meromorphic function field on such a complex manifold is always generated by the sections of a certain holomorhic line bundle L. Then the sections of the real holomorphic bundle $\mathbf{L} \otimes \sigma^* \bar{\bar{\mathbf{L}}}$ also generates the function field. As real holomorphic line bundles on such a twistor space can be proved to be generated by the anticanonical bundle, $L \otimes \sigma^* \bar{L}$ is a positive power of the anticanonical bundle. However, when the scalar curvature is negative, a Bochner type argument, using the twistor operator, shows that any positive power of the anticanonical bundle has no nontrivial sections. When the scalar curvature is equal to zero, this argument is refined to show that sections of any power of the anticanonical bundle cannot generate the function field. Examples of Moishezon twistor spaces are the twistor spaces of a self-dual structure on the connected-sum of two and three copies of the complex projective plane [P_1, P_2]. They are the small resolution of the intersection of two quadrics in $\mathbb{CP}^5$ with four nodes and the double covering of $\mathbb{CP}^3$ branched along a quartic with 13 nodes respectively. More generally, when the 4-fold is simply connected, if the algebraic dimension of a compact twistor space is positive, one can prove that the self-dual 4-fold is either the K3-surface with Calabi-Yau metric and opposite orientation or homeomorphic to the connected-sum of complex projective planes with a self-dual conformal class containing metrics with positive scalar curvature [P_4, V]. The motivation of this generalization is that the construction of Donaldson and Friedman shows that the connected-sum of projective planes always admits such a family of self-dual conformal class. One might want to know if there are special elements in this family. As the conformal structure is encoded in the holomorphic structure of the twistor space, one may want to compute a certain algebraic invariant on the twistor space. Therefore, the algebraic dimension is one of the invariant that we want to know. For example, a generic twistor space over the connected-sum of four $\mathbb{CP}^2$ has algebraic dimension one [DF]. Can it jump or not?.

§4 In [S_1], Salamon showed that there is a generalization of the Ward correspondence of instantons and holomorphic bundles on the twistor spaces. Essentially, the Ward correspondence states that there is a one-to-one correspondence between SU(2) bundles $\mathbf{E}$ with self-dual connection on a self-dual manifolds and real holomorphic rank 2 bundle $\pi^*\mathbf{E}$ on the twistor space that is trivial on every real twistor line. A self-dual SU(2) connection is usually called an instanton. It is well-known that instanton is the minimum of the Yang-Mills functional:

$$\mathcal{YM}(\nabla) = \int_X \| F_\nabla \|^2 \mathrm{vol}$$

$$= -\int_X \mathrm{Tr}(F_\nabla \wedge * F_\nabla) \quad ,$$

where F_∇ is the curvature of the connection of ∇. In fact, the curvature can be splitted into a direct summands:

$$F_\nabla = F^+ + F^-,$$

where F^+ and F^- satisfy the self-dual and anti-self-dual equations respectivley:

$$*F^+ = F^+ , \quad *F^- = -F^- .$$

Then $\mathcal{YM}(\nabla) = \| F^+ \|^2 + \| F^- \|^2$.

This functional is bounded below by $8\pi^2|k|$ where $-k$ is the second Chern number of the SU(2) bundle. And

$$8\pi^2 k = \| F^+ \|^2 - \| F^- \|^2.$$

Salamon generalizes the Ward correspondence algebraically in the following way: by definition, the principal bundle of frame on a quaternionic manifold X can be reduced to $GL(n,\mathbb{H})Sp(1)$. If E is the complex rank 2n dimension representation of $GL(n, \mathbb{H})$ given by the inclusion in $GL(2n, \mathbb{C})$ and H is the irreducible rank 2 representation of Sp(1), then the bundle of 2-forms on X is associated to the frame bundle by the representation $\Lambda^2(E\otimes H)$ [S2]. It has a decomposition:

$$\Lambda^2(E\otimes H) \cong S^2E\otimes\Lambda^2H \oplus \Lambda^2E\otimes S^2H$$
$$\cong S^2E \oplus \Lambda^2E\otimes S^2H. \quad (4.1)$$

If the manifold is actually a quaternionic Kähler manifold, then E has an invariant symplectic form and $\Lambda^2(E\otimes H)$ is further decomposed into

$$\Lambda^2(E\otimes H) \cong S^2E \oplus S^2H \oplus \Lambda_0^2E\otimes S^2H, \quad (4.2)$$

where Λ_0^2E is the orthogonal compliment of the symplectic form on E in Λ^2E. Then the bundle of 2-forms has a corresponding decomposition. Of course, when n=1, Λ^2E is one dimensional and the above sum is simply the sum of the complexification of self-dual and anti-self-dual 2-forms. Salamon called a vector bundle **V** on a quaternionic manifold quaternionic if it admits a $GL(p,\mathbb{H})$ structure and a connection whose curvature 2-form is in the component S^2E. And he found that for such bundle, as in the Ward coorespondence, $\pi^*\mathbf{V}$ on the twistor space is a holomorphic bundle. Examples are the tangent bundles on quaternionic Kähler manifolds [S_1].

On the other hand, the Yang-Mills functional on quaternionic Kähler manifold can be modified [GP] as follows:

$$\mathcal{YM}_c(\nabla) \equiv \tfrac{1}{2}\int_X \; [\, \| F_\nabla \|^2 + c^2 \, \| F_\nabla \wedge \Omega^{n-1} \|^2 \,] \quad ,$$

where c is a real number and

$$\Omega \equiv \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3 ,$$

when ω_1, ω_2, ω_3 forms an orthogonal frame with length $\sqrt{2}n$ on the bundle associated to the representation S^2H. Part of the reasons that we choose this modification is that this functional has a lower bound that depends only on the topology of the bundle **V** and the geometry of X [GP]:

$$c\,(8\pi^2 \int_X p_1(\mathbf{V}) \wedge \Omega^{n-1}) \leq \mathcal{YM}_c(\nabla)$$

with equality if and only if

$$*F_\nabla = c\, F_\nabla \wedge \Omega^{n-1}. \tag{4.3}$$

When n=1, this equation is reduced to

$$*F_\nabla = c\, F_\nabla .$$

And because the $*$-operator on 2-forms over four dimensional manifold is an involution, c must be either 1 or -1. Equation (4.3) is reduced to the self-dual and anti-self-dual equations (4.1). Note that although the Yang-Mills functional is modified so that solutions to equation (4.3) are absolute minima or maxima, the solutions are also critical points of the standard Yang-Mills functional because the Euler-Lagrange equations of the Yang-Mills functional are $d^\nabla F_\nabla = 0$

and $d^{\nabla} * F_{\nabla} = 0$. And these equations are satisfied because the fundamental 4-form Ω is parallel and F_{∇} satisfied the Bianchi identity.

Now the obvious question is how the Salamon's notion of quaternionic bundle is related to this generalization of Yang-Mills functional. It turns out that if F is a 2-form such that

$$* F = - \frac{1}{(2n-1)!} F \wedge \Omega^{n-1} \tag{4.4}$$

then F is in the component S^2E. To express this equation pointwisely, one may choose an orthonormal basis $\{\omega_0^j, \omega_1^j, \omega_2^j, \omega_3^j;\ j=1,...,n.\}$ on the cotangent bundle over a point on X. If

$$\omega_1 \equiv \Sigma_1^n (\omega_0^j \wedge \omega_1^j + \omega_2^j \wedge \omega_3^j),$$

$$\omega_2 \equiv \Sigma_1^n (\omega_0^j \wedge \omega_2^j - \omega_1^j \wedge \omega_3^j),$$

$$\omega_3 \equiv \Sigma_1^n (\omega_0^j \wedge \omega_3^j + \omega_1^j \wedge \omega_2^j),$$

then $\{ \omega_1, \omega_2, \omega_3 \}$ is an orthogonal frame on S^2H with lenght $\sqrt{2n}$.

and $\Omega \equiv \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$. When $F_{\binom{i}{\alpha}\binom{j}{\beta}}$ is the coefficient of a 2-form F with respect to the frame

$$\{ \omega_\alpha^i \wedge \omega_\beta^j : \alpha, \beta=0, 1, 2, 3; \ i, j=1,\dots,n \},$$

then F satisfies equation (4.4) if and only if

$$F_{\binom{i}{0}\binom{j}{1}} = -F_{\binom{i}{2}\binom{j}{3}}, \qquad F_{\binom{i}{0}\binom{j}{2}} = F_{\binom{i}{1}\binom{j}{3}}, \qquad F_{\binom{i}{0}\binom{j}{3}} = - F_{\binom{i}{1}\binom{j}{2}},$$

$$F_{\binom{i}{\alpha}\binom{j}{\alpha}} = F_{\binom{i}{\beta}\binom{j}{\beta}}, \qquad F_{\binom{i}{\alpha}\binom{j}{\beta}} = F_{\binom{j}{\alpha}\binom{i}{\beta}}, \qquad \forall\ \alpha, \beta, i, j. \tag{4.5}$$

Going through the identification:

$$\Lambda^2 T^* \cong \Lambda^2(E\otimes H) \cong S^2E \oplus S^2H \oplus \Lambda_0^2 E\otimes S^2H,$$

one can check that such an F is in the component S^2E. Counting the degree of freedom, one can conclude that any F in S^2E satisfies these equations. In particalur, the connection defining a quaternionic vector bundle is a minimun of the generalized Yang-Mills functional. This observation justifies Salamon's approach to generalize the concept of instantons on quaternionic Kähler manifold.

Moreover, the construction of S^2E does not depend on the choice of a quaternionic Kähler metric within a given quaternionic structure, equation (4.4) is independent of the choice of metric so long as there is one quaternionin Kähler metric within the given structure. This is the analogue in the four dimensional case that the self-dual and anti-self-dual equations are conformally invariant. Similarly, one can show that a 2-form F is in the component S^2H if and only if it satisfies the following generalization of self-dual equation:

$$* F = \frac{6n}{(2n+1)!} F \wedge \Omega^{n-1}$$

However, except when n=1, this equation depends on the choice of a metric as the decomposition (4.2) does.

To find examples of solution to this pair of equations and see the relation between equation (4.4) and other topics. Let's consider the projection [H2,H3]

$$p: \mathbb{R}^4 \otimes \mathbb{R}^n \to \mathbb{R}^3 \otimes \mathbb{R}^n$$

given by

$$(x_0^i, x_1^i, x_2^i, x_3^i) \mapsto (x_1^i, x_2^i, x_3^i),$$

where $1 \le i \le n$. Suppose that **V** is a vector bundle on $\mathbb{R}^4 \otimes \mathbb{R}^n$ with connection ∇' invariant under the translation in the coordinates x_0^i, then $\mathbf{V} = p^* \widetilde{\mathbf{V}}$ for a bundle $\widetilde{\mathbf{V}}$ on $\mathbb{R}^3 \otimes \mathbb{R}^n$ and

$$\nabla' = p^* \nabla + \Sigma_i \Phi^i dx_0^i,$$

where Φ^i is a section of the adjoint bundle of $\widetilde{\mathbf{V}}$. One can check that when the curvature F of ∇' satisfies the equation (4.4), or equivalently equation (4.5), then

$$F_{X^i_\alpha X^j_\beta} = \Sigma_\gamma \varepsilon_{\alpha\beta\gamma} \nabla_{X^i_\gamma} \Phi^j + \tfrac{1}{2}\delta_{\alpha\beta}[\Phi^i, \Phi^j]$$

and $\nabla_{X^i_\alpha}\Phi^j = \nabla_{X^j_\beta}\Phi^i$, where $\alpha,\beta=1,2,3$; $i,j=1,\ldots n$.

This is the generalization of Bogomolny equation found in [PP_1]. If the adjoint bundle of $\widetilde{\mathbf{V}}$ is abelien, this pair of equation is reduced to

$$F_{X^i_\alpha X^j_\beta} = \Sigma_\gamma \varepsilon_{\alpha\beta\gamma} \nabla_{X^i_\gamma} \Phi^j \text{ and } \nabla_{X^i_\alpha}\Phi^j = \nabla_{X^j_\beta}\Phi^i.$$

It was shown [HK, PP_1] that coefficients of a hyperkähler metric on a 4n dimensional space with n commuting Killing field preserving the hyperkähler structure will determine a solution to this pair of equation.

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