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Generalized contact structures

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Abstract

We study integrability of generalized almost contact structures, and find conditions under which the main associated maximal isotropic vector bundles form Lie bialgebroids. These conditions differentiate the concept of generalized contact structures from a counterpart of generalized complex structures on odd-dimensional manifolds. We name the latter strong generalized contact structures. Using a Boothby–Wang construction bridging symplectic structures and contact structures, we find examples to demonstrate that, within the category of generalized contact structures, classical contact structures have non-trivial deformations. Using deformation theory of Lie bialgebroids, we construct new families of strong generalized contact structures on the three-dimensional Heisenberg group and its cocompact quotients.

1. *Introduction*

The theory of generalized complex structures is a geometric framework unifying both complex structures and symplectic structures [**11**, **13**]. It is applicable only to even-dimensional manifolds. A key feature of this theory is to allow deformation between complex and symplectic structures. There are indeed non-trivial examples of such phenomenon on compact manifolds [**11**, **20**]. This phenomenon is a departure from Moser's theorem on the rigidity of symplectic structures with respect to diffeomorphisms [**19**].

For centuries, symplectic structures and contact structures have often been studied in parallel, beginning as frameworks for classical mechanics [**1**, **16**]. For instance, both symplectic and contact structures have 'standard' local models. Moser's theorem has its counterpart for contact structures [**10**]. Boothby and Wang showed that, when a contact structure is represented by a 'regular' 1-form, the underlying manifold is foliated and the leaf space has a symplectic structure. Conversely, the total space of an SO(2)-bundle on a symplectic manifold with the given symplectic form as a curvature form has a contact structure [**3**]. From the viewpoint of G-structures, contact structures are also related to complex structures. This perspective leads to Sasaki's introduction of normal almost contact structures on odd-dimensional manifolds [**23**].

While many of the similarities between symplectic and contact structures have been emphasized, and relations between complex and contact structures have been developed, we often ignore a fundamental distinction of contact structures, namely, from a G-structure perspective, symplectic structures and complex structures are integrable. The pseudogroup of their local models is transitive [**5**, **6**]. Although the pseudogroup of contact transformations remains transitive [**4**], contact structures are not integrable G-structures.

In this article, we continue a recent search to develop an analog of generalized complex structures on odd-dimensional manifolds [**14**, **21**, **24**]. In [**14**], the second author and her collaborator developed a concept called generalized almost contact structures. Its development is based on the theory of Dirac structures and 1-jet bundles of the underlying manifolds. In [**24**],

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Vaisman developed the concept of 'generalized almost contact structures of codimension k'. He also introduced and studied generalized F-structures and CRF-structures in [25]. We focus on the codimension one case in Vaisman's development, and simply call it a 'generalized almost contact structure' in this article (see Definition 2.1).

In a recent paper, we investigated the integrability of such structures from Sasaki's perspective [21]. However, the theory of generalized complex structures is developed in the context of Lie bialgebroids [11, 17]. This concept requires the splitting of the complexification of the direct sum of the tangent bundle TN and the cotangent bundle T^*N over an even-dimensional manifold N into the direct sum of two maximally isotropic sub-bundles L and L^*. Integrability is in terms of the closedness of the spaces of sections of these bundles under the Courant bracket [11]. The pair L and L^* forms a Lie bialgebroid. The core of this paper is to analyze generalized almost contact structures in such a context.

Given a generalized almost contact structure $\mathcal{J}$ on a manifold M, readers will see that the bundle $(TM \oplus T^*M)_{\mathbb{C}}$ splits into the direct sum of two maximally isotropic subbundles, L and its dual L^*. Unlike generalized almost complex structures on even-dimensional manifolds, L^* is not complex-conjugate linearly isomorphic to L. Therefore, one has to analyze L and L^* individually. In Section 2.3, we identify the obstruction for $\Gamma(L^*)$ to be closed under the assumption that the space $\Gamma(L)$ of sections of L is closed under the Courant bracket. The main result of this section is Theorem 2.7.

If the structure $\mathcal{J}$ is defined by a classical contact 1-form, then we show that the space $\Gamma(L)$ is closed under the Courant bracket (see Section 3.1). However, by analyzing the local model of a contact structure, we find that the obstruction for $\Gamma(L^*)$ to be closed under the Courant bracket does not vanish (see Proposition 3.1). Therefore, we define a generalized (integrable) contact structure to be a generalized almost contact structure whose corresponding $\Gamma(L)$ is closed under the Courant bracket, while $\Gamma(L^*)$ is not necessarily closed (see Definition 2.4). When both $\Gamma(L)$ and $\Gamma(L^*)$ are closed, we consider the given structure $\mathcal{J}$ as a natural counterpart of a generalized complex structure on an odd-dimensional manifold, and refer to $\mathcal{J}$ as a strong generalized contact structure (see Definition 2.8). The distinction between generic generalized contact structures and the strong version could be conceived as an extension of the fact that classical contact structures are not integrable G-structures.

We find examples for these new concepts from two different sources. One of these sources is from classical geometry, and another is through deformation theory.

The classical analogs of symplectic and complex structures were discovered nearly half a century ago. When studying infinitesimal automorphisms of symplectic structures, Libermann developed the concept of an almost cosymplectic structure [15]. As a G-structure, it is a reduction of the structure group of a $(2n+1)$-dimensional manifold from $\mathrm{GL}(2n+1,\mathbb{R})$ to $\{1\} \times \mathrm{Sp}(n,\mathbb{R})$ [8, 15]. In terms of tensors, it is equivalent to the choice of a 1-form η and a 2-form θ such that $\eta \wedge \theta^n \neq 0$ at every point of the manifold. An almost cosymplectic structure (η, θ) is a cosymplectic structure if it is an integrable G-structure. It is equivalent to both η and θ being closed forms [15]. By choosing $\theta = d\eta$, it is immediate that a contact 1-form η yields an almost cosymplectic structure, but it is *not* integrable as $d\eta$ is non-zero everywhere.

Treating an almost complex structure on a $2n$-dimensional manifold N as a reduction of the principal bundle of frames from $\mathrm{GL}(2n,\mathbb{R})$ to $\mathrm{GL}(n,\mathbb{C})$, we obtain a $(1,1)$-tensor J on the manifold N such that $J \circ J = -\mathbb{I}$. An almost contact structure on an odd-dimensional manifold M is a triple (F, η, φ) consisting of a vector field F, a 1-form η and a $(1,1)$-tensor φ such that $\varphi^2 = -\mathbb{I} + F \otimes \eta$. This triple could be used to naturally define an almost complex structure on the cone over M [2, 4, 23]. When this almost complex structure is integrable, the triple is called a normal almost contact structure. Readers are warned of the very unfortunate historical fact that, without an auxiliary geometric object, a contact 1-form does not naturally define an almost contact structure, and nor does a normal almost contact structure [4].

In Sections 3.2 and 3.3, it is shown that cosymplectic structures and normal almost contact structures, respectively, are examples of strong generalized contact structures. Since such structures are associated with Lie bialgebroid theory, we are able to apply a deformation theory as developed in [**17**] to generate new and non-classical examples (see Section 4.2).

As noted in Section 3.1, classical contact structures are examples of non-strong generalized contact structures. We illustrate this point on SU(2) in Section 4.1. To find non-trivial new examples, we apply a Boothby–Wang construction on an SO(2)-bundle on the Kodaira surface N [**3**]. We first note that there exists an analytic family of generalized complex structures J_t with parameter t such that J_0 is a classical complex structure on a Kodaira surface N, and J_1 is a symplectic structure on N [**20**]. Following [**3**], we construct a family of generalized contact structures $\mathcal{J}_t$ on a principal SO(2)-bundle M over the Kodaira surface such that $\mathcal{J}_1$ is associated to a classical contact 1-form on M. The quotient of the structure $\mathcal{J}_t$ on M by the fundamental vector field of the principal bundle yields the family J_t on N. This example explicitly illustrates that classical contact 1-forms, when conceived as generalized contact structures, are not necessarily rigid. It is a departure from Gray's theorem [**10**].

2. *Generalized contact structures*

For a manifold M of any dimension, consider the vector bundle $TM \oplus T^*M \to M$. Its space of sections is endowed with two natural $\mathbb{R}$-bilinear operations.

(1) A symmetric bilinear form $\langle \cdot, \cdot \rangle$ is defined by

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\iota_X \beta + \iota_Y \alpha). \tag{2.1}$$

(2) The Courant bracket is given by

$$[\![X + \alpha, Y + \beta]\!] = [X, Y] + \mathcal{L}_X \beta - \mathcal{L}_Y \alpha - \frac{1}{2} d(\iota_X \beta - \iota_Y \alpha). \tag{2.2}$$

We adopt the following notation: $(\pi^\sharp \alpha)(\beta) = \pi(\alpha, \beta)$ and $Y(\theta^\flat X) = \theta(X, Y)$ for any 1-forms α and β, 2-form θ, bivector field π, and vector fields X and Y.

The bundle $TM \oplus T^*M$ with the non-degenerate pairing $\langle -, - \rangle$ in (2.1) and Courant bracket (2.2) above form a fundamental example of a *Courant algebroid* [**7**, **17**]. The natural projection ρ from the direct sum to the summand TM is called the anchor map.

We consider the complexified bundles, and complex-linearly extend the symmetric bilinear form and the Courant bracket to obtain complex Courant algebroids.

2.1. *Generalized almost contact structures*

Definition 2.1 [**21**, **24**]. *A generalized almost contact pair* on a smooth odd-dimensional manifold M consists of a bundle endomorphism Φ from $TM \oplus T^*M$ to itself and a section $F + \eta$ of $TM \oplus T^*M$ such that $\Phi + \Phi^* = 0$, $\eta(F) = 1$, $\Phi(F) = 0$, $\Phi(\eta) = 0$, and $\Phi \circ \Phi = -\mathbb{I} + F \odot \eta$.

Here $F \odot \eta$ acts on $TM \oplus T^*M$ as a symmetric bundle endomorphism, that is, when $X + \alpha$ is a section of $TM \oplus T^*M$, then we define

$$(F \odot \eta)(X + \alpha) := \eta(X)F + \alpha(F)\eta.$$

The pair of tensors $(\Phi, F + \eta)$ is equivalent to another pair $(\Phi', F' + \eta')$ if there exists a function f without a zero on the manifold M such that

$$\Phi' = \Phi, \quad \eta' = f\eta, \quad F' = \frac{1}{f}F. \tag{2.3}$$

Definition 2.2. *A generalized almost contact structure* on M is an equivalence class of a pair $(\Phi, F + \eta)$.

In terms of components, a generalized almost contact structure is given by the following equivalence class of tensorial objects: $\mathcal{J} = (F, \eta, \pi, \theta, \varphi)$, where F is a vector field, η is a 1-form, π is a bivector field, θ is a 2-form, and φ is a (1,1)-tensor. They are subjected to the following relations:

$$\theta^\flat \varphi = \varphi^* \theta^\flat, \quad \varphi \pi^\sharp = \pi^\sharp \varphi^*, \tag{2.4}$$

$$\varphi^2 + \pi^\sharp \theta^\flat = -\mathbb{I} + F \otimes \eta, \quad (\varphi^*)^2 + \theta^\flat \pi^\sharp = -\mathbb{I} + \eta \otimes F, \tag{2.5}$$

$$\eta \circ \varphi = \varphi^* \eta = 0, \quad \eta \circ \pi^\sharp = \pi^\sharp \eta = 0, \quad \iota_F \varphi = 0, \quad \iota_F \theta = 0, \quad \iota_F \eta = 1. \tag{2.6}$$

The bundle map $\Phi : TM \oplus T^*M \to TM \oplus T^*M$ is given by

$$\Phi = \begin{pmatrix} \varphi & \pi^\sharp \\ \theta^\flat & -\varphi^* \end{pmatrix}. \tag{2.7}$$

2.2. *The associated complex vector sub-bundles*

Consider the above bundle map Φ. It has one real eigenvalue, namely, 0. The corresponding eigenbundle is trivialized by F and η, and we denote these bundles by L_F and L_η, respectively. Let $\ker \eta$ be the distribution on the manifold M defined by the point-wise kernel of the 1-form η. Similarly, $\ker F$ is the sub-bundle of T^*M defined by the point-wise kernel of the vector field F with respect to its evaluation on differential 1-forms. On the complexified bundle $(TM \oplus T^*M)_{\mathbb{C}}$, we have that Φ has three eigenvalues, namely, '0', '$+i$', and '$-i$'. Let us define

$$E^{(1,0)} = \{e - i\ \Phi(e) \mid e \in \ker \eta \oplus \ker F\},$$
$$E^{(0,1)} = \{e + i\ \Phi(e) \mid e \in \ker \eta \oplus \ker F\}.$$

Then $L_F \oplus L_\eta$ is the 0-eigenbundle, $E^{(1,0)}$ is the $+i$-eigenbundle, and $E^{(0,1)}$ is the $-i$-eigenbundle. We have a natural splitting $(TM \oplus T^*M)_{\mathbb{C}} = L_F \oplus L_\eta \oplus E^{(1,0)} \oplus E^{(0,1)}$. It is apparent that this decomposition does not depend on any choice of representatives within an equivalence class of generalized contact forms. The choice of a trivialization of a real sub-bundle L_F in TM, a trivialization of its dual L_η in T^*M, and the subsequent choice of $E^{(1,0)} \oplus E^{(0,1)}$ determines a generalized contact pair.

In the subsequent analysis, the following four different complex vector bundles will play different roles:

$$\begin{aligned} L &:= L_F \oplus E^{(1,0)}, \quad \overline{L} := L_F \oplus E^{(0,1)}, \\ L^* &:= L_\eta \oplus E^{(0,1)}, \quad \overline{L}^* := L_\eta \oplus E^{(1,0)}. \end{aligned} \tag{2.8}$$

As L_F is the complexification of a real line bundle, its conjugation is itself. Therefore, the complex conjugation map sends L to $\overline{L}$. On the other hand, through the symmetric pairing (2.1), L^* is complex-linearly isomorphic to the dual of L. All of these bundles are independent of the choice of representatives of a generalized almost contact structure.

Lemma 2.3. *The bundles $E^{(1,0)}$, $E^{(0,1)}$, L, $\overline{L}$, L^*, and $\overline{L}^*$ are isotropic with respect to the symmetric pairing $\langle -, - \rangle$.*

Proof. Suppose that $X + \alpha$ is a section of $\ker \eta \oplus \ker F$. Then

$$\Phi(X + \alpha) = \varphi(X) + \pi^\sharp(\alpha) + \theta^\flat(X) - \varphi^*(\alpha).$$

By constraints (2.6), $\Phi(X + \alpha)$ is again a section of $\ker \eta \oplus \ker F$. Therefore,

$$\langle F, \Phi(X + \alpha) \rangle = 0 \quad \text{and} \quad \langle \eta, \Phi(X + \alpha) \rangle = 0. \tag{2.9}$$

If both $X + \alpha$ and $Y + \beta$ are sections of $\ker \eta \oplus \ker F$, then

$$\begin{aligned}&\langle X + \alpha - i\Phi(X + \alpha), Y + \beta - i\Phi(Y + \beta)\rangle \\ &\quad = \langle X, \beta\rangle - i\langle X, \theta^\flat(Y) - \varphi^*(\beta)\rangle - i\langle \alpha, \varphi(Y) + \pi^\sharp(\beta)\rangle \\ &\qquad + \langle Y, \alpha\rangle - i\langle Y, \theta^\flat(X) - \varphi^*(\alpha)\rangle - i\langle \beta, \varphi(X) + \pi^\sharp(\alpha)\rangle \\ &\qquad - \langle \varphi(X) + \pi^\sharp(\alpha), \theta^\flat(Y) - \varphi^*(\beta)\rangle - \langle \varphi(Y) + \pi^\sharp(\beta), \theta^\flat(X) - \varphi^*(\alpha)\rangle.\end{aligned}$$

Since θ and π are skew-symmetric, the above is reduced to

$$\langle X, \beta\rangle + \langle \alpha, Y\rangle - \langle \varphi(X) + \pi^\sharp(\alpha), \theta^\flat(Y) - \varphi^*(\beta)\rangle - \langle \varphi(Y) + \pi^\sharp(\beta), \theta^\flat(X) - \varphi^*(\alpha)\rangle.$$

By constraints (2.5), it is further reduced to

$$-\langle \varphi(X), \theta^\flat(Y)\rangle + \langle \pi^\sharp(\alpha), \varphi^*(\beta)\rangle - \langle \varphi(Y), \theta^\flat(X)\rangle + \langle \pi^\sharp(\beta), \varphi^*(\alpha)\rangle.$$

By (2.4), this expression is equal to zero. It follows that $E^{(1,0)}$ is isotropic.

Taking the complex conjugation in the above computation, we find that $E^{(0,1)}$ is also isotropic. By (2.9), the pairings between sections of L_F or L_η with those of $E^{(1,0)} \oplus E^{(0,1)}$ are always equal to zero. Therefore $L = L_F \oplus E^{(1,0)}$ is isotropic. A similar computation shows that $L^* = L_\eta \oplus E^{(0,1)}$ is isotropic. □

Definition 2.4. Given a generalized almost contact structure, if the space $\Gamma(L)$ of sections of the associated bundle L is closed under the Courant bracket, then the generalized almost contact structure is simply called a *generalized contact structure*.

Since L_F is a rank-1 bundle, it is apparent that $[\![\Gamma(L_F), \Gamma(L_F)]\!] \subseteq \Gamma(L_F)$. Therefore, the non-trivial conditions for $[\![\Gamma(L), \Gamma(L)]\!] \subseteq \Gamma(L)$ are due to the following two inclusions:

$$[\![\Gamma(L_F), \Gamma(E^{(1,0)})]\!] \subseteq \Gamma(L_F \oplus E^{(1,0)}),$$
$$[\![\Gamma(E^{(1,0)}), \Gamma(E^{(1,0)})]\!] \subseteq \Gamma(L_F \oplus E^{(1,0)}).$$

As a consequence of Lemma 2.3, all four bundles given in (2.8) are *maximally* isotropic with respect to the pairing (2.1) in $(TM \oplus T^*M)_{\mathbb{C}}$. Combined with the concept given in Definition 2.4, the definition of 'Dirac structures' [**7**], and the definition of a 'quasi'-Lie bialgebroid in [**22**], we have the following corollary.

Corollary 2.5. *When $\mathcal{J} = (F, \eta, \pi, \theta, \varphi)$ represents a generalized contact structure, the associated bundle L is a Dirac structure. In addition, the bundle L^* is a transversal isotropic complement of L in the Courant algebroid $((TM \oplus T^*M)_{\mathbb{C}}, \langle -, -\rangle, [\![-, -]\!])$. In other words, the pair L and L^* is a quasi-Lie bialgebroid.*

2.3. *Obstruction to integrability of the dual bundle L^**

A lack of natural isomorphism between L and L^* means that, when $\Gamma(L)$ is closed under the Courant bracket, $\Gamma(L^*)$ is not necessarily closed. It is a major departure from the theory of generalized complex structures on even-dimensional manifolds. In this section, with [**11**, **17**] as our key references, we examine the obstruction for both L and L^* being closed.

Recall [**18**] that a complex Lie algebroid on a manifold M is a complex vector bundle V together with a bundle map $\rho : V \to TM_{\mathbb{C}}$, called the anchor map, and a bracket $[\![-, -]\!]$ on the space of sections of V such that, for any sections s_1, s_2, and s_3 of V and any smooth function f on M, the following hold:

(1) $[\![s_1, s_2]\!] = -[\![s_2, s_1]\!]$;
(2) $[\![[\![s_1, s_2]\!], s_3]\!] + [\![[\![s_2, s_3]\!], s_1]\!] + [\![[\![s_3, s_1]\!], s_2]\!] = 0$;

(3) $[\![s_1, fs_2]\!] = f[\![s_1, s_2]\!] + (\rho(s_1)f)s_2$;
(4) $\rho([\![s_1, s_2]\!]) = [\rho(s_1), \rho(s_2)]$.

In our previous discussion on generalized contact structures, we focused on the bundle L. In terms of a Lie algebroid, the inclusion of L in $(TM \oplus T^*M)_{\mathbb{C}}$ followed by the natural projection onto the first summand is an anchor map. When $\Gamma(L)$ is closed under the Courant bracket, the restriction of the Courant bracket to L completes the construction of a Lie algebroid structure on L.

Given the natural anchor map ρ on $(TM \oplus T^*M)_{\mathbb{C}}$ and assuming that $\Gamma(L)$ is closed, the next issue is whether the space $\Gamma(L^*)$ of sections of L^* is closed under the Courant bracket.

It is known that the obstruction for $\Gamma(L^*)$ to be closed is an alternating form on L^* [**17**, Lemma 3.2]. It could be regarded as a section of $\wedge^3(L^*)^* \cong \wedge^3 L$. It was called the 'Nijenhuis operator' in [**11**, Proposition 3.16]. It is denoted by Nij. Its relation with the Jacobi identity was explicitly given in [**11**]. Since L^* is maximally isotropic in $(TM \oplus T^*M)_{\mathbb{C}}$ with respect to the symmetric pairing, the obstruction for $\Gamma(L^*)$ being closed with respect to the Courant bracket is the restriction of Nij on $\Gamma(L^*)$ [**11**, Proposition 3.27]. To be precise, for any three sections v_0, v_1, and v_2 of $\Gamma(L^*)$, we have

$$\mathrm{Nij}(v_0, v_1, v_2) = \frac{1}{3}(\langle [\![v_0, v_1]\!], v_2\rangle + \langle [\![v_1, v_2]\!], v_0\rangle + \langle [\![v_2, v_0]\!], v_1\rangle). \tag{2.10}$$

To compute Nij, recall that L_F is rank 1 and $L = L_F \oplus E^{(1,0)}$. Therefore, Nij has two components due to the decomposition

$$\wedge^3 L = (L_F \oplus \wedge^2 E^{(1,0)}) \oplus \wedge^3 E^{(1,0)}.$$

Now assume that $\Gamma(L)$ is closed under the Courant bracket. By conjugation,

$$[\![\Gamma(E^{(0,1)}), \Gamma(E^{(0,1)})]\!] \subseteq \Gamma(L_F \oplus E^{(0,1)}) = \Gamma(\overline{L}).$$

Since $\overline{L}$ is isotropic, it follows that $\langle E^{(0,1)}, L_F \oplus E^{(0,1)}\rangle = 0$. Therefore, if v_0, v_1, and v_2 are all sections of $E^{(0,1)}$, then $\mathrm{Nij}(v_0, v_1, v_2) = 0$. Hence, up to a permutation, Nij is uniquely determined by

$$\mathrm{Nij}(v_0, v_1, \eta) = \frac{1}{3}(\langle [\![v_0, v_1]\!], \eta\rangle + \langle [\![v_1, \eta]\!], v_0\rangle + \langle [\![\eta, v_0]\!], v_1\rangle), \tag{2.11}$$

where v_0 and v_1 are sections of $E^{(0,1)}$.

PROPOSITION 2.6. *The Nijenhuis operator* Nij *for a generalized contact structure* $\mathcal{J} = (F, \eta, \pi, \theta, \varphi)$ *is*

$$\mathrm{Nij} = -\frac{1}{2} F \wedge (\rho^* d\eta)^{(2,0)}, \tag{2.12}$$

where $(\rho^* d\eta)^{(2,0)}$ *is the* $\wedge^2 E^{(1,0)}$*-component of the pullback of* $d\eta$ *via the anchor map* $\rho : L^* \to TM$.

Proof. Suppose that X and Y are sections of $\ker \eta$, and that α and β are sections of $\ker F$. Let $v_0 = X + \alpha + i\Phi(X + \alpha)$ and $v_1 = Y + \beta + i\Phi(Y + \beta)$. In terms of the components of Φ, we have

$$v_0 = X + \alpha + i\pi^\sharp \alpha + i\varphi X + i\theta^\flat X - i\varphi^* \alpha, \quad \text{and} \quad \rho(v_0) = X + i\varphi X + i\pi^\sharp \alpha.$$

There is a similar expression for $Y + \beta + i\Phi(Y + \beta)$. Note that the Courant bracket between any two 1-forms is equal to zero, and the space of 1-forms is isotropic with respect to the symmetric bilinear pairing (2.1). It follows that

$$\begin{aligned}\mathrm{Nij}(v_0, v_1, \eta) = &\ \frac{1}{3}(\langle [\![\rho(v_0), \rho(v_1)]\!], \eta\rangle + \langle [\![\rho(v_1), \eta]\!], \rho(v_0)\rangle + \langle [\![\eta, \rho(v_0)]\!], \rho(v_1)\rangle)\\ = &\ \frac{1}{6}(\eta([\![\rho(v_0), \rho(v_1)]\!]) + (\mathcal{L}_{\rho(v_1)}\eta)\rho(v_0) - (\mathcal{L}_{\rho(v_0)}\eta)\rho(v_1)).\end{aligned}$$

Since $\eta(X + i\varphi X + i\pi^\sharp\alpha) = \eta(Y + i\varphi Y + i\pi^\sharp\beta) = 0$, the above is equal to

$$\begin{aligned}&\frac{1}{6}(\eta([\![\rho(v_0), \rho(v_1)]\!]) + (\iota_{\rho(v_1)}d\eta)\rho(v_0) - (\iota_{\rho(v_0)}d\eta)\rho(v_1))\\ &\quad = \frac{1}{6}(-d\eta(\rho(v_0), \rho(v_1)) + (\iota_{\rho(v_1)}d\eta)\rho(v_0) - (\iota_{\rho(v_0)}d\eta)\rho(v_1))\\ &\quad = -\frac{1}{2}d\eta(\rho(v_0), \rho(v_1)).\end{aligned}$$

Therefore, Nij is given as claimed. □

Note that, if f is a function without a zero such that $F' = (1/f)F$ and $\eta' = f\eta$, then, on $(\ker\eta \oplus \ker F)_{\mathbb{C}}$, we have $d\eta' = f d\eta$. Therefore, the equality in (2.12) is independent of the choice of representative tensors within a given generalized contact structure.

Suppose that L and L^* are both Lie algebroids. Let d_L be the Lie algebroid differential associated to the bracket on L. It acts on the space of sections of $\wedge^k(L^*)$. Similarly, we have a differential d_{L^*} associated to the Lie algebroid structure of L^*. It acts on sections of $\wedge^k L$. Since both L and L^* inherit the bracket from the Courant bracket on $(TM \oplus T^*M)_{\mathbb{C}}$, and they are dual to each other with respect to the symmetric pairing (2.1), they together naturally form a Lie bialgebroid [**17**]. To summarize our discussion so far, we have the following theorem.

THEOREM 2.7. *Let $\mathcal{J} = (F, \eta, \pi, \theta, \varphi)$ represent an (integrable) generalized contact structure. The pair L and L^* forms a Lie bialgebroid if and only if $d\eta$ is of type $(1,1)$ with respect to the map Φ on $(\ker\eta \oplus \ker F)_{\mathbb{C}}$.*

Proof. Since η is a real 1-form, it follows that $d\eta$ is a real 2-form. Therefore, $(\rho^* d\eta)^{(2,0)}$ is the complex conjugation of $(\rho^* d\eta)^{(0,2)}$. Therefore, $(\rho^* d\eta)^{(0,2)} = 0$ if and only if $(\rho^* d\eta)^{(2,0)} = 0$. □

The above analysis indicates a special class of objects among generalized contact structures.

DEFINITION 2.8. An almost generalized contact structure is called a *strong generalized contact structure* if both $\Gamma(L)$ and $\Gamma(L^*)$ are closed under the Courant bracket.

2.4. *Integrability of the associated complex sub-bundles*

Suppose that $\mathcal{J} = (F, \eta, \pi, \theta, \varphi)$ represents a strong generalized contact structure. By complex conjugation, the closedness of $\Gamma(L^*)$ is equivalent to the closedness of $\Gamma(\overline{L}^*)$. Since $L = L_F \oplus E^{(1,0)}$ and $\overline{L}^* = L_\eta \oplus E^{(1,0)}$, it follows that

$$[\![\Gamma(E^{(1,0)}), \Gamma(E^{(1,0)})]\!] \subseteq \Gamma(L_F \oplus E^{(1,0)}) \cap \Gamma(L_\eta \oplus E^{(1,0)}). \tag{2.13}$$

This inclusion implies that

$$[\![\Gamma(E^{(1,0)}), \Gamma(E^{(1,0)})]\!] \subseteq \Gamma(E^{(1,0)}) \tag{2.14}$$

and that the corresponding statement with a complex conjugation holds.

With respect to the symmetric non-degenerate bilinear pairing (2.1), the dual of $E^{(1,0)}$ is its conjugate bundle $E^{(0,1)}$. In this section, we focus on the structures of these two bundles.

Our issue now is whether the pair $E^{(1,0)}$ and $E^{(0,1)}$ forms a Lie bialgebroid. Since both bundles are Lie algebroids, the only point of concern is whether there is a natural compatibility between Lie algebroid differentials and the Courant brackets.

By natural compatibility, we mean treating the bundles $E^{(1,0)}$ and $E^{(0,1)}$ as sub-bundles of $(TM \oplus T^*M)_{\mathbb{C}}$ with the Courant bracket (2.2). While $(TM \oplus T^*M)_{\mathbb{C}}$ is a Courant algebroid, the direct sum $E^{(1,0)} \oplus E^{(0,1)}$ may fail to be one because the bracket between sections of $E^{(1,0)}$ and $E^{(0,1)}$ with respect to the Courant bracket on $(TM \oplus T^*M)_{\mathbb{C}}$ may not be a section of $E^{(1,0)} \oplus E^{(0,1)}$.

Let ω be a section of $E^{(1,0)}$ and let $\overline{\sigma}$ be a section of $E^{(0,1)}$. Suppose that the pair $E^{(1,0)}$ and $E^{(0,1)}$ forms a Lie bialgebroid. Then $[\![\omega, \overline{\sigma}]\!]$ is a section of $E^{(1,0)} \oplus E^{(0,1)}$. In particular, $\eta(\rho[\![\omega, \overline{\sigma}]\!]) = \eta([\rho(\omega), \rho(\overline{\sigma})]) = 0$. By the definitions of $E^{(1,0)}$ and $E^{(0,1)}$, we have $\eta(\rho(\omega)) = 0$ and $\eta(\rho(\overline{\sigma})) = 0$. Therefore,

$$d\eta(\rho(\omega), \rho(\overline{\sigma})) = -\eta([\rho(\omega), \rho(\overline{\sigma})]) = 0.$$

In other words, $(\rho^* d\eta)^{(1,1)}$ vanishes identically on $\ker \eta$. Since we initially assumed that $d\eta$ is of type $(1,1)$, it follows that $d\eta$ vanishes identically on $\ker \eta$.

Conversely, let d_E and $d_{\overline{E}}$ be the Lie algebroid differentials for $E^{(1,0)}$ and $E^{(0,1)}$, respectively. The differential d_E is the composition of an inclusion map, the differential d_L, and a projection. More precisely, given $\Gamma(\wedge^k L^*) = \Gamma(\wedge^k (L_\eta \oplus E^{(0,1)}))$ for all k, the differential d_E is given by

$$d_E : \Gamma(\wedge^k E^{(0,1)}) \hookrightarrow \Gamma(\wedge^k L^*) \xrightarrow{d_L} \Gamma(\wedge^{k+1} L^*) \xrightarrow{p} \Gamma(\wedge^{k+1} E^{(0,1)}),$$

where the map

$$p : \Gamma(\wedge^{k+1} L^*) = \Gamma((L_\eta \otimes \wedge^k E^{(0,1)}) \oplus (\wedge^{k+1} E^{(0,1)})) \longrightarrow \Gamma(\wedge^{k+1} E^{(0,1)})$$

is a natural projection, that is, $d_E \overline{\alpha} = p(d_L \overline{\alpha})$ for each section $\overline{\alpha}$ of $\wedge^k E^{(0,1)}$.

To check whether the pair $E^{(1,0)}$ and $E^{(0,1)}$ forms a Lie bialgebroid, we need to verify that

$$d_{\overline{E}} [\![\omega_1, \omega_2]\!] = [\![d_{\overline{E}} \omega_1, \omega_2]\!] + [\![\omega_1, d_{\overline{E}} \omega_2]\!] \tag{2.15}$$

for any pair of sections ω_1 and ω_2 of the bundle $E^{(1,0)}$. Since the pair L and L^* forms a Lie bialgebroid, it follows that

$$d_{L^*} [\![\omega_1, \omega_2]\!] = [\![d_{L^*} \omega_1, \omega_2]\!] + [\![\omega_1, d_{L^*} \omega_2]\!]. \tag{2.16}$$

When Ω is a section of $\wedge^2 L = (L_F \otimes E^{(1,0)}) \oplus \wedge^2 E^{(1,0)}$, let Ω^F be the first component of Ω in this decomposition. Then the above identity becomes

$$\begin{aligned} &(d_{L^*} [\![\omega_1, \omega_2]\!])^F + d_{\overline{E}} [\![\omega_1, \omega_2]\!] \\ &\quad = [\![(d_{L^*} \omega_1)^F, \omega_2]\!] + [\![d_{\overline{E}} \omega_1, \omega_2]\!] + [\![\omega_1, (d_{L^*} \omega_2)^F]\!] + [\![\omega_1, d_{\overline{E}} \omega_2]\!]. \end{aligned} \tag{2.17}$$

To calculate $(d_{L^*} \omega)^F$ for any section ω of $E^{(1,0)}$, let $\overline{\sigma}$ be any section of $E^{(0,1)}$. Then, by definition,

$$(d_{L^*} \omega)(\eta, \overline{\sigma}) = \rho(\eta) \langle \omega, \overline{\sigma} \rangle - \rho(\overline{\sigma}) \langle \omega, \eta \rangle - \langle \omega, [\![\eta, \overline{\sigma}]\!] \rangle.$$

Since $\rho(\eta) = 0$ and ω is a section of $\ker \eta \oplus \ker F$, the above is reduced to

$$\begin{aligned} \langle \omega, \mathcal{L}_{\rho(\overline{\sigma})} \eta \rangle = \langle \rho(\omega), \mathcal{L}_{\rho(\overline{\sigma})} \eta \rangle = \langle \rho(\omega), \iota_{\rho(\overline{\sigma})} d\eta \rangle &= \frac{1}{2} d\eta(\rho(\overline{\sigma}), \rho(\omega)) \\ = -\frac{1}{2} (\rho^* d\eta)(\omega, \overline{\sigma}) &= -\frac{1}{2} (\iota_\omega \rho^* d\eta)(\overline{\sigma}). \end{aligned}$$

It follows that, as a section of $L_F \otimes E^{(1,0)} \subset \wedge^2 L$, we have

$$(d_{L^*} \omega)^F = -\frac{1}{2} F \wedge (\iota_\omega \rho^* d\eta)^{(1,0)}. \tag{2.18}$$

Suppose that $(\rho^* d\eta)^{(1,1)} = 0$. Then $d\eta(\rho(\overline{\sigma}), \rho(\omega)) = 0$. It means that $(d_{L^*}\omega)(\eta, \overline{\sigma}) = 0$ for all sections ω of $E^{1,0}$ and $\overline{\sigma}$ of $E^{0,1}$. Therefore, identity (2.17) is equivalent to identity (2.15). It means that the pair $E^{(1,0)}$ and $E^{(0,1)}$ forms a Lie bialgebroid.

Note that, since we initially assumed that $(\rho^* d\eta)^{(2,0)} = 0$, the assumption that $(\rho^* d\eta)^{(1,1)} = 0$ is equivalent to $\rho^* d\eta = 0$ on $\ker \eta$. The next proposition follows.

PROPOSITION 2.9. *Let $\mathcal{J} = (F, \eta, \pi, \theta, \varphi)$ represent a strong generalized contact structure. The pair $E^{(1,0)}$ and $E^{(0,1)}$ with the induced Courant bracket is a Lie bialgebroid if and only if $d\eta$ vanishes identically on* $\ker \eta$.

3. *Some classical geometry in odd dimensions*

3.1. *Contact structures*

Suppose that M is a $(2n+1)$-dimensional manifold with a 1-form η such that $\eta \wedge (d\eta)^n$ is non-zero everywhere. Then the 1-form η is a contact 1-form.

To make an almost generalized contact structure associated to the contact 1-form η, let $\theta = d\eta$. Then the map

$$\flat(X) := \iota_X \theta - \eta(X)\eta \tag{3.1}$$

is an isomorphism from the tangent bundle to the cotangent bundle. In particular, there is a unique vector field F such that $\iota_F \eta = 1$ and $\iota_F d\eta = 0$. This vector field is known as the Reeb field of the contact form η. Let us define a bivector field π by

$$\pi(\alpha, \beta) := \theta(\flat^{-1}(\alpha), \flat^{-1}(\beta)). \tag{3.2}$$

We choose $\varphi = 0$, and

$$\Phi = \begin{pmatrix} 0 & \pi^\sharp \\ \theta^\flat & 0 \end{pmatrix}. \tag{3.3}$$

Then, the map Φ, the Reeb field F, and the contact form η define an almost generalized contact structure.

As the differential forms η and θ are invariant with respect to the Reeb field F, the map Φ is also invariant. Therefore,

$$\mathcal{L}_F \eta = 0, \quad \mathcal{L}_F \theta = 0, \quad \mathcal{L}_F \Phi = 0.$$

Next, we examine the properties of the associated bundles L and L^*.

By the Darboux theorem [**12**], in a neighborhood of any point on M, there exist local coordinates $(x_1, y_1, \dots, x_n, y_n, z)$ such that

$$\eta = dz - \sum_{j=1}^{n} y_j dx_j. \tag{3.4}$$

The Reeb field is naturally $F = \partial/\partial z$, and $\theta = d\eta = \sum_{j=1}^{n} dx_j \wedge dy_j$. Let

$$X_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial z}, \quad Y_j = \frac{\partial}{\partial y_j}.$$

Then $\{X_j, Y_j, F\}$ forms a moving frame on the given coordinate, and $\{dx_j, dy_j, \eta\}$ forms a coframe. By construction (3.1), we have

$$\flat(X_j) = dy_j, \quad \flat(Y_j) = -dx_j, \quad \flat(F) = -\eta.$$

By (3.2), we have

$$\pi = \sum_{j=1}^{n} X_j \wedge Y_j. \tag{3.5}$$

It follows that $\Phi(\eta) = 0$ and that

$$\Phi(X_j) = \theta^\flat(X_j) = dy_j, \quad \Phi(Y_j) = \theta^\flat(Y_j) = -dx_j,$$
$$\Phi(dx_j) = \pi^\sharp(dx_j) = Y_j, \quad \Phi(dy_j) = \pi^\sharp(dy_j) = -X_j.$$

Then a local frame for $E^{(1,0)}$ is $\{X_j - idy_j, Y_j + idx_j\}$. A local frame for $E^{(0,1)}$ is $\{X_j + idy_j, Y_j - idx_j\}$. Since

$$[\![F, X_j - idy_j]\!] = 0, \quad [\![F, Y_j + idx_j]\!] = 0, \tag{3.6}$$
$$[\![X_j - idy_j, Y_j + idx_j]\!] = [\![X_j, Y_j]\!] = -F, \tag{3.7}$$

the spaces of sections of the bundles $L = L_F \oplus E^{(1,0)}$ and $\overline{L} = L_F \oplus E^{(0,1)}$ are closed under the Courant bracket. It explicitly shows that L and $\overline{L}$ are Lie algebroids. On the other hand,

$$[\![X_j - idy_j, \eta]\!] = \iota_{X_j} d\eta = dy_j, \quad [\![Y_j + idx_j, \eta]\!] = \iota_{Y_j} d\eta = -dx_j.$$

Therefore, $\Gamma(L^*) = \Gamma(L_\eta \oplus E^{(1,0)})$ and $\Gamma(\overline{L}^*) = \Gamma(L_\eta \oplus E^{(0,1)})$ are not closed under the Courant bracket.

As the obstruction to closedness is $F \wedge (\rho^* d\eta)^{2,0}$, we could also find it through the type decomposition of $\rho^* d\eta$. Given $\theta = d\eta = \sum_{j=1}^{n} dx_j \wedge dy_j$ and the map Φ above, it is straightforward to find that

$$(\rho^* d\eta)^{2,0} = \frac{1}{4}\sum_{j=1}^{n}(dx_j - iY_j) \wedge (dy_j + iX_j),$$
$$(\rho^* d\eta)^{0,2} = \frac{1}{4}\sum_{j=1}^{n}(dx_j + iY_j) \wedge (dy_j - iX_j),$$
$$(\rho^* d\eta)^{1,1} = \frac{1}{2}\sum_{j=1}^{n}(dx_j \wedge dy_j + X_j \wedge Y_j) = \frac{1}{2}(d\eta + \pi).$$

In particular, the obstruction for closedness of $\Gamma(L^*)$ does not vanish anywhere.

PROPOSITION 3.1. *Let $\mathcal{J} = (F, \eta, \pi, \theta, \varphi)$ represent a generalized contact structure associated to a classical contact 1-form η. Then the corresponding bundles L and $\overline{L}$ are Dirac structures. The bundles L^* and $\overline{L}^*$ are never Dirac structures. In particular, the pair L and L^* is not a Lie bialgebroid.*

3.2. *Almost cosymplectic structures*

An almost cosymplectic structure consists of a 1-form η and a 2-form θ such that $\eta \wedge \theta^n \neq 0$ at every point of the manifold. Given this condition, the map formally given in (3.1) is again an isomorphism. Therefore, there exists a unique vector field F such that $\eta(F) = 1$ and $\theta(F) = 0$. These tensors determine an almost generalized contact structure by the matrix Φ. It is formally given in (3.3).

If both η and θ are closed, then we address the pair (η, θ), a cosymplectic structure without qualification. Next, we investigate the integrability of the generalized almost contact structure associated to a cosymplectic structure (η, θ). Since $\iota_F\theta = 0$, for any section X of $\ker\eta$, we have $[\![F, X - i\iota_X\theta]\!] = [F, X] - i\mathcal{L}_F\iota_X\theta$. Since θ is closed and $\iota_F\theta = 0$, it follows that $\mathcal{L}_F\theta = 0$.

As $\mathcal{L}_F\iota_X\theta - \iota_X\mathcal{L}_F\theta = \iota_{[F,X]}\theta$, it follows that

$$[\![F, X - i\iota_X\theta]\!] = [F, X] - i\iota_{[F,X]}\theta.$$

If X and Y are sections of $\ker\eta$, then

$$\begin{aligned}[\![X - i\iota_X\theta, Y - i\iota_Y\theta]\!] &= [X, Y] - i(\mathcal{L}_X\iota_Y\theta - \mathcal{L}_Y\iota_X\theta) + id(\iota_X\iota_Y\theta)\\ &= [X, Y] - i\iota_{[X,Y]}\theta - i(\iota_Y\mathcal{L}_X\theta - \mathcal{L}_Y\iota_X\theta - d\iota_X\iota_Y\theta).\end{aligned}$$

This expression is equal to $[X, Y] - i\iota_{[X,Y]}\theta$ due to $d\theta = 0$ and the identity $\mathcal{L}_X = d\circ\iota_X + \iota_X\circ d$.

Through the isomorphism $\flat$, the computation above also shows, for any sections α and β of $\ker F$, that $[\![\alpha - i\iota_\alpha\pi, \beta - i\iota_\beta\pi]\!]$ is a section of $E^{(1,0)}$. Similarly, $[\![X - i\iota_X\theta, \beta - i\iota_\beta\pi]\!]$ is a section of $E^{(1,0)}$ whenever both $X - i\iota_X\theta$ and $\beta - i\iota_\beta\pi$ are. Therefore, $\Gamma(E^{(1,0)})$ is closed under the Courant bracket.

It follows that $\Gamma(L)$ is closed under the Courant bracket. In addition, since $d\eta = 0$, by Theorem 2.7, Definition 2.8, and Proposition 3.1, we have the following observation.

PROPOSITION 3.2. *If $\mathcal{J}$ represents a generalized almost contact structure associated to a classical cosymplectic structure, then it is a strong generalized contact structure. Moreover, the pairs of bundles (L, L^*) and $(E^{(1,0)}, E^{(0,1)})$ with respect to the induced Courant bracket are both Lie bialgebroids.*

3.3. *Almost contact structures*

Suppose that M is a $(2n+1)$-dimensional manifold with a vector field F, a 1-form η, and a type (1,1)-tensor φ satisfying

$$\varphi^2 = -\mathbb{I} + \eta\otimes F \quad \text{and} \quad \eta(F) = 1. \tag{3.8}$$

Then the triple (φ, F, η) is a called an *almost contact structure* [**23**].

Associated to any almost contact structure, we have an almost generalized contact structure by setting

$$\Phi = \begin{pmatrix}\varphi & 0\\ 0 & -\varphi^*\end{pmatrix} \tag{3.9}$$

with the given vector field F and 1-form η.

An almost contact structure is a 'normal almost contact structure' [**2**] if

$$\mathcal{N}_\varphi = -F\otimes d\eta, \quad \mathcal{L}_F\varphi = 0, \quad \text{and} \quad \mathcal{L}_F\eta = 0, \tag{3.10}$$

where, by definition,

$$\mathcal{N}_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi([\varphi X, Y] + [X, \varphi Y]) \tag{3.11}$$

for any vector fields X and Y. Note that the equations (3.10) imply that, if s is a section of $E^{(1,0)}$, then $[\![F, s]\!]$ is again a section of $E^{(1,0)}$.

Since $\mathcal{N}_\varphi = -F\otimes d\eta$, it follows that, for any vector fields X and Y, we have

$$-\,d\eta(X, Y)F = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi([\varphi X, Y] + [X, \varphi Y]). \tag{3.12}$$

In particular, this identity holds when the vector fields are sections of the bundle $\ker\eta$. In such a case, by applying η on both sides of this identity, we find that

$$\eta([\varphi X, \varphi Y]) = \eta(\mathcal{N}_\varphi(X, Y)) = -d\eta(X, Y). \tag{3.13}$$

As φX and φY are also sections of $\ker\eta$, the above identity implies that

$$d\eta(\varphi X, \varphi Y) = d\eta(X, Y) \tag{3.14}$$

for any sections X and Y in $\ker\eta$. Therefore, the restriction of $d\eta$ on $\ker\eta$ is of type (1,1) with respect to Φ. For future reference, we present this observation as the following lemma.

LEMMA 3.3. *Suppose that (F,η,φ) is a normal almost contact structure. Then $\rho^*d\eta$ is a section of $E^{(1,0)}\otimes E^{(0,1)}$.*

Now, for any sections X and Y of $\ker\eta$, it follows that, due to the first identity in (3.8), we have

$$\begin{aligned}[\![X-i\varphi X, Y-i\varphi Y]\!] &= [X,Y]-[\varphi X,\varphi Y]-i([\varphi X,Y]+[X,\varphi Y])\\ &= [X,Y]+\varphi^2[\varphi X,\varphi Y]-i([\varphi X,Y]+[X,\varphi Y])-\eta([\varphi X,\varphi Y])F. \quad (3.15)\end{aligned}$$

Since φX and φY are sections of $\ker\eta$, we have $\eta([\varphi X,\varphi Y])=-d\eta(\varphi X,\varphi Y)$. Applying formula (3.12) to the pair of vector fields φX and φY, and observing that $\varphi^2X=-X$ and $\varphi^2Y=-Y$, we get

$$-d\eta(\varphi X,\varphi Y)F=[X,Y]+\varphi^2[\varphi X,\varphi Y]+\varphi([X,\varphi Y]+[\varphi X,Y]).$$

Given (3.8) and (3.14), we have that the right-hand side of (3.15) is equal to

$$\begin{aligned}&-\varphi([\varphi X,Y]+[X,\varphi Y])-i([\varphi X,Y]+[X,\varphi Y])\\ &\quad=-\varphi([\varphi X,Y]+[X,\varphi Y])+i\varphi^2([\varphi X,Y]+[X,\varphi Y])-i\eta([\varphi X,Y]+[X,\varphi Y])F\\ &\quad=-\varphi([\varphi X,Y]+[X,\varphi Y])+i\varphi^2([\varphi X,Y]+[X,\varphi Y]).\end{aligned}$$

Therefore,

$$[\![X-i\varphi X,Y-i\varphi Y]\!]=-\varphi([\varphi X,Y]+[X,\varphi Y])+i\varphi^2([\varphi X,Y]+[X,\varphi Y]).$$

Since $-\varphi([\varphi X,Y]+[X,\varphi Y])$ is a section of $\ker\eta$, the above tensor is a section of $E^{(1,0)}$.

Next, suppose that X is a section of $\ker\eta$ and that β is a section of $\ker F$. By definition,

$$\begin{aligned}[\![X-i\varphi X,\beta+i\varphi^*\beta]\!]&=\mathcal{L}_{(X-i\varphi X)}(\beta+i\varphi^*\beta)-\frac{1}{2}d\iota_{(X-i\varphi X)}(\beta+i\varphi^*\beta)\\ &=\mathcal{L}_{(X-i\varphi X)}(\beta+i\varphi^*\beta)\\ &=\mathcal{L}_X\beta+\mathcal{L}_{(\varphi X)}(\varphi^*\beta)+i(\mathcal{L}_X(\varphi^*\beta)-\mathcal{L}_{(\varphi X)}\beta).\end{aligned}$$

Evaluating the real part of the above expression on the Reeb field, with standard tensor calculus and (3.8), we find that it is equal to $\eta([F,X])\beta(F)$. Since β is a section of $\ker F$, the real part of the above expression is a section of $\ker F$. Next, due to the transpose of the first formula in (3.8), we have

$$\begin{aligned}\varphi^*(\mathcal{L}_X\beta+\mathcal{L}_{\varphi X}(\varphi^*\beta))&=\mathcal{L}_X(\varphi^*\beta)-(\mathcal{L}_X\varphi)^*\beta+\mathcal{L}_{(\varphi X)}((\varphi^*)^2\beta)-(\mathcal{L}_{(\varphi X)}\varphi)^*(\varphi^*\beta)\\ &=\mathcal{L}_X(\varphi^*\beta)-(\mathcal{L}_X\varphi)^*\beta+\mathcal{L}_{(\varphi X)}(-\beta+\beta(F)\eta)-(\mathcal{L}_{(\varphi X)}\varphi)^*(\varphi^*\beta)\\ &=\mathcal{L}_X(\varphi^*\beta)-\mathcal{L}_{(\varphi X)}\beta-(\mathcal{L}_X\varphi)^*\beta-(\mathcal{L}_{(\varphi X)}\varphi)^*(\varphi^*\beta).\end{aligned}$$

We claim that $(\mathcal{L}_X\varphi)^*\beta+(\mathcal{L}_{(\varphi X)}\varphi)^*(\varphi^*\beta)=0$. To verify this, we let A be any vector field, and we have

$$\begin{aligned}((\mathcal{L}_X\varphi)^*\beta)A+((\mathcal{L}_{(\varphi X)}\varphi)^*(\varphi^*\beta))A&=\beta((\mathcal{L}_X\varphi)A+\varphi\circ(\mathcal{L}_{(\varphi X)}\varphi)A)\\ &=\beta([X,\varphi A]-\varphi[X,A]+\varphi[\varphi X,\varphi A]-\varphi^2[\varphi X,A])\\ &=\beta([X,\varphi A]-\varphi[X,A]+\varphi[\varphi X,\varphi A]\\ &\qquad+[\varphi X,A]-\eta([\varphi X,A])F)\\ &=\beta(\varphi\mathcal{N}_\varphi(X,A)).\end{aligned}$$

By (3.10), we have $\mathcal{N}_\varphi(X,A)=-d\eta(X,A)F$. Since $\varphi(F)=0$, it follows that $\varphi\mathcal{N}_\varphi(X,A)=0$.

Since the Courant bracket between two 1-forms is always equal to zero, we could now conclude that the Courant bracket between two sections of $E^{(1,0)}$ is again a section of $E^{(1,0)}$. Since the bundle $E^{(1,0)}$ is F-invariant, the bundle L is closed with respect to the Courant bracket.

Finally, formula (3.14) shows that $(\rho^* d\eta)^{(2,0)} = 0$. Therefore, L^* is closed with respect to the Courant bracket. By Theorem 2.7, we have the following proposition.

PROPOSITION 3.4. *If $\mathcal{J}$ represents a generalized almost contact structure associated to a classical normal almost contact structure on an odd-dimensional manifold M, then it is a strong generalized contact structure.*

4. *Examples of strong generalized contact structures*

4.1. *Structures on* SU(2)

On the Lie algebra $\mathfrak{su}(2)$, choose a basis X_1, X_2, and X_3 and a dual basis σ^1, σ^2, and σ^3 such that

$$[X_1, X_2] = -X_3, \quad d\sigma^1 = \sigma^2 \wedge \sigma^3, \tag{4.1}$$

and similar conditions with cyclic permutations of the indices $\{1, 2, 3\}$ hold.

4.1.1. *Normal contact structures on* SU(2). To construct a classical normal almost contact structure, one simply takes

$$\eta = \sigma^3, \quad F = X_3, \quad \varphi = X_2 \otimes \sigma^1 - X_1 \otimes \sigma^2. \tag{4.2}$$

Then, we have

$$\varphi^* = \varphi = -\sigma^2 \otimes X_1 + \sigma^1 \otimes X_2. \tag{4.3}$$

Therefore,

$$\Phi(X_1) = \varphi(X_1) = X_2, \quad \Phi(X_2) = \varphi(X_2) = -X_1,$$
$$\Phi(\sigma^1) = -\varphi^*(\sigma^1) = \sigma^2, \quad \Phi(\sigma^2) = -\varphi^*(\sigma^2) = -\sigma^1.$$

The bundles L and L^* are globally trivialized. As modules over the space of smooth functions, we have

$$\Gamma(L) = \Gamma(L_F \oplus E^{1,0}) = \left\langle X_3, \frac{1}{\sqrt{2}}(X_1 - iX_2), \frac{1}{\sqrt{2}}(\sigma^1 - i\sigma^2) \right\rangle,$$
$$\Gamma(L^*) = \Gamma(L_\eta \oplus E^{0,1}) = \left\langle \sigma^3, \frac{1}{\sqrt{2}}(X_1 + iX_2), \frac{1}{\sqrt{2}}(\sigma^1 + i\sigma^2) \right\rangle.$$

It is now an elementary computation to verify that the structure equations for the Lie algebroids L and L^* are given by

$$\left[\!\left[X_3, \frac{1}{\sqrt{2}}(X_1 - iX_2) \right]\!\right] = -\frac{i}{\sqrt{2}}(X_1 - iX_2),$$
$$\left[\!\left[X_3, \frac{1}{\sqrt{2}}(\sigma^1 - i\sigma^2) \right]\!\right] = -\frac{i}{\sqrt{2}}(\sigma^1 - i\sigma^2),$$
$$\left[\!\left[\sigma^3, \frac{1}{\sqrt{2}}(X_1 + iX_2) \right]\!\right] = \frac{i}{\sqrt{2}}(\sigma^1 + i\sigma^2).$$

On the other hand, we have

$$[\![X_1 - iX_2, X_1 + iX_2]\!] = -2iX_3. \tag{4.4}$$

It demonstrates that $\Gamma(E^{1,0} \oplus E^{0,1})$ is not closed under the Courant bracket. In other words, with respect to the induced Courant bracket, $E^{1,0} \oplus E^{0,1}$ is not a Courant algebroid [**17**]. It follows that the pair $E^{1,0}$ and $E^{0,1}$, with respect to the induced Courant bracket, does not form a Lie bialgebroid. This example demonstrates that Proposition 3.2 for cosymplectic structures cannot be extended to normal almost contact structures, or strong generalized contact structures in general.

4.1.2. *Contact structures on* SU(2). An obvious contact structure on SU(2) is given by $\eta = \sigma^3$. In such a case, we have

$$F = X_3, \quad \theta = d\sigma^3 = \sigma^1 \wedge \sigma^2, \quad \pi = X_1 \wedge X_2. \tag{4.5}$$

With $\varphi = 0$, the restriction of Φ on $\ker \sigma^3 \oplus \ker X_3$ is determined by

$$\Phi(X_1) = \sigma^2, \quad \Phi(X_2) = -\sigma^1, \quad \Phi(\sigma^1) = X_2, \quad \Phi(\sigma^2) = -X_1. \tag{4.6}$$

Therefore,

$$L = \langle X_3, X_1 - i\sigma^2, X_2 + i\sigma^1 \rangle, \quad L^* = \langle \sigma^3, X_1 + i\sigma^2, X_2 - i\sigma^1 \rangle. \tag{4.7}$$

Taking the Courant brackets, we find that

$$[\![X_3, X_1 - i\sigma^2]\!] = -(X_2 + i\sigma^1), \quad [\![X_3, X_2 + i\sigma^1]\!] = X_1 - i\sigma^2,$$
$$[\![X_1 - i\sigma^2, X_2 + i\sigma^1]\!] = -X_3 = [\![X_1 + i\sigma^2, X_2 - i\sigma^1]\!],$$
$$[\![\sigma^3, X_1 + i\sigma^2]\!] = -\sigma^2, \quad [\![\sigma^3, X_2 - i\sigma^1]\!] = \sigma^1.$$

This example reaffirms that L forms a Lie algebroid, while L^* fails to be one.

4.2. *Structures on the three-dimensional Heisenberg group*

On the three-dimensional Heisenberg group H_3, we choose a basis $\{X_1, X_2, X_3\}$ for its algebra $\mathfrak{h}_3$ so that $[X_1, X_2] = -X_3$. Let $\{\alpha^1, \alpha^2, \alpha^3\}$ be a dual frame. Then $d\alpha^3 = \alpha^1 \wedge \alpha^2$.

4.2.1. *Cosymplectic structure on* H_3. For any real numbers a and b, choose

$$\eta = \alpha^1 \quad \text{and} \quad \theta = \alpha^2 \wedge \alpha^3 + a\alpha^1 \wedge \alpha^2 + b\alpha^1 \wedge \alpha^3. \tag{4.8}$$

They together define a cosymplectic structure. The Reeb field is $F = X_1 - bX_2 + aX_3$. Since

$$\flat(X_1) = a\alpha^2 + b\alpha^3 - \alpha^1, \quad \flat(X_2) = \alpha^3 - a\alpha^2, \quad \flat(X_3) = -b\alpha^1 - \alpha^1,$$

it follows that $\pi = X_2 \wedge X_3$ and $\varphi = 0$. It can be shown that $\ker F = \langle \alpha^2 + b\alpha^1, \alpha_3 - a\alpha^1 \rangle$ and $\ker \eta = \langle X_2, X_3 \rangle$. Since

$$\Phi(X_2) = \alpha^3 - a\alpha^1, \quad \Phi(X_3) = -\alpha^2 - b\alpha^1,$$

we obtain global sections to trivialize the bundles L and L^* as follows:

$$L = \langle X_1 - bX_2 + aX_3, X_2 - i\alpha^3 + ia\alpha^1, X_3 + i\alpha^2 + ib\alpha^1 \rangle,$$
$$L^* = \langle \alpha^1, X_2 + i\alpha^3 - ia\alpha^1, X_3 - i\alpha^2 - ib\alpha^1 \rangle.$$

Since the Courant brackets between X_3, α^1, and α^2 and any element among $X_1, X_2, X_3, \alpha^1, \alpha^2$, and α^3 are equal to zero, the restriction of the Courant bracket on L^* is identically equal to zero. The restriction on L is determined by a single non-trivial equation, namely,

$$[X_1 - bX_2 + aX_3, X_2 - i\alpha^3 + ia\alpha^1] = -(X_3 + i\alpha^2 + ib\alpha^1).$$

4.2.2. *New examples on* H_3. For $t = rc + irs$, where $c = \cos\vartheta$ and $s = \sin\vartheta$ for some real number ϑ, we define

$$\varphi_t := \frac{2rc}{1-r^2}(X_2 \otimes \alpha^2 + X_3 \otimes \alpha^3),$$
$$\theta_t := \frac{r^2 - 2rs + 1}{1-r^2}\alpha^2 \wedge \alpha^3, \quad \pi_t = \frac{r^2 + 2rs + 1}{1-r^2} X_2 \wedge X_3.$$

Now, as given in (2.7), we define

$$\Phi_t := \begin{pmatrix} \varphi_t & \pi_t^\sharp \\ \theta_t^\flat & -\varphi_t^* \end{pmatrix}.$$

Then $\mathcal{J}_t := (F, \eta, \pi_t, \theta_t, \varphi_t)$ is a family of generalized almost contact structures. The corresponding bundle L_t and its conjugate $\overline{L}_t$ are trivialized as follows:

$$\begin{aligned} L_t &= \langle X_1, (X_2 - i\alpha^3) - i\Phi_t(X_2 - i\alpha^3), (X_3 + i\alpha^2) - i\Phi_t(X_3 + i\alpha^2)\rangle \\ &= \langle X_1, (1+rs)X_2 + rc\alpha^3 - i(1-rs)\alpha^3 - ircX_2, \\ &\qquad (1+rs)X_3 - rc\alpha^2 + i(1-rs)\alpha^2 - ircX_3\rangle, \\ L_t^* &= \langle \alpha^1, (\alpha^2 + iX_3) + i\Phi_t(\alpha^2 + iX_3), (\alpha^3 - iX_2) + i\Phi_t(\alpha^3 - iX_2)\rangle \\ &= \langle \alpha^1, (1-rs)\alpha^2 - rcX_3 + i(1+rs)X_3 - irc\alpha^2, \\ &\qquad (1-rs)\alpha^3 + rcX_2 - i(1+rs)X_2 - irc\alpha^3\rangle. \end{aligned}$$

Since the Courant brackets between X_3, α^1, and α^2 and any element among $X_1, X_2, X_3, \alpha^1, \alpha^2$, and α^3 are equal to zero, it is straightforward to check that the restriction of the Courant bracket to $\Gamma(L_t^*)$ is trivial. On $\Gamma(L_t)$, the sole non-trivial bracket is due to

$$\begin{aligned} &[\![X_1, (1+rs)X_2 + rc\alpha^3 - i(1-rs)\alpha^3 - ircX_2]\!] \\ &\quad = -((1+rs)X_3 - rc\alpha^2 + i(1-rs)\alpha^2 - ircX_3). \end{aligned}$$

Therefore, $\mathcal{J}_t$ is an analytic family of strong generalized contact structures.

In this family, there are two apparent subfamilies, determined by $|t|^2 = r^2 < 1$ and $|t|^2 = r^2 > 1$.

When $t = 0$, we recover the strong generalized contact structure determined by a cosymplectic structure as given in (4.8) with $a = b = 0$.

When $r \neq 0$ and $\cos\vartheta \neq 0$, the strong generalized contact structure is no longer given by a classical cosymplectic structure. Since the polynomials $r^2 - 2r\sin\vartheta + 1$ and $r^2 + 2r\sin\vartheta + 1$ do not have zeros for any ϑ, the family does not contain any classical almost contact structures either.

When $r \to \infty$, we recover the cosymplectic structure with 1-form $\eta = \alpha^1$ and 2-form $\theta_\infty = -\alpha^2 \wedge \alpha^3$.

4.2.3. *Deformation of cosymplectic structures.* Recall from Proposition 3.2 that the pair of bundles $E^{1,0}$ and $E^{0,1}$ forms a Lie bialgebroid with respect to the restriction of the Courant bracket when they are determined by a classical cosymplectic structure. Let the Lie algebroid differential for the former be denoted by d_E and that for the latter be denoted by $d_{\overline{E}}$. Suppose that Γ is a section of $\wedge^2 E^{0,1}$. It is also treated as a section of $\mathrm{Hom}(E^{1,0}, E^{0,1})$. If it satisfies the Maurer–Cartan equation

$$d_E\Gamma + \frac{1}{2}[\![\Gamma, \Gamma]\!] = 0, \tag{4.9}$$

then the graph of Γ is a deformation of the Lie algebroid $E^{1,0}$ [**17**]. Let us denote it by $E_\Gamma^{1,0}$.

Since $E^{0,1}$ is a complex conjugation of $E^{1,0}$, the graph of $\overline{\Gamma}$ determines a deformation of $E^{0,1}$ and $E_{\overline{\Gamma}}^{0,1}$. As it is clear that $E_{\overline{\Gamma}}^{0,1}$ is isomorphic to the complex conjugation of $E_\Gamma^{1,0}$, we obtain a deformation of a cosymplectic structure through strong generalized contact structures, with the Reeb field F and the 1-form η unperturbed.

In the example of Section 4.2.2, the restrictions of the Courant bracket on both $E^{1,0}$ and $E^{0,1}$ are trivial. It follows that the section

$$\Gamma = (\alpha^2 + iX_3) \wedge (\alpha^3 - iX_2) \tag{4.10}$$

solves the Maurer–Cartan equation (4.9). Therefore, we obtain deformations. To recover the family of strong generalized contact structures on H_3 in example in Section 4.2.2, one simply takes $r(\cos\vartheta + i\sin\vartheta)\Gamma$ to generate new examples.

5. *Examples of generalized contact structures*

It is well known that if η is a regular contact 1-form on a compact manifold M, then M is a principal circle bundle over a smooth manifold N such that η is a connection 1-form. Here N is the space of leaves of the foliation of the Reeb field F for the contact 1-form η. Moreover, there exists a symplectic form ω on N such that the curvature form of η is given by $d\eta = -p^*\omega$, where $p : M \to N$ is the quotient map [**3**]. A converse construction of contact structures on any principal SO(2)-bundle, whose characteristic class is a symplectic form, is easily developed through the identity $d\eta = -p^*\omega$. In this section, we illustrate how these constructions could be done for generalized contact structures, at least in the case when the manifolds involved are Lie groups and the geometry is invariant. Finally, we produce a non-trivial family of generalized contact structures, with a classical contact 1-form in the family; thereby, we demonstrate that classical contact structures have a deformation that is in the category of generalized contact structures, and away from classical objects. It leads to a departure from Gray's theorem that, up to diffeomorphisms, contact structures on compact manifolds do not have a non-trivial deformation among classical contact 1-forms [**10**].

5.1. *On central extensions of even-dimensional Lie groups*

Suppose that H is a real Lie group with an invariant symplectic form ω. Let $\mathfrak{h}$ be the Lie algebra of H, with Lie bracket $\{- \cdot -\}$. Denote by $\mathfrak{c}$ a one-dimensional real vector space. Let F be a non-zero vector in $\mathfrak{c}$. On the space $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{c}$, we next define a new Lie bracket $[-,-]$ on $\mathfrak{g}$ as follows. For any X and Y in $\mathfrak{h}$, we have

$$[X, Y] := \{X \cdot Y\} + \omega(X, Y)F, \quad \text{and} \quad [X, F] = 0. \tag{5.1}$$

To check that $[-,-]$ is indeed a Lie bracket, one needs only to check that the Jacobi identity is satisfied by a triple of elements in $\mathfrak{h}$. It turns out to be a consequence of $d\omega = 0$ and $\{- \cdot -\}$ satisfying the Jacobi identity. This construction makes $\mathfrak{g}$ a central extension of $\mathfrak{h}$ by $\mathfrak{c}$.

Elements in $\mathfrak{h}^*$ are extended to be elements in $\mathfrak{g}^*$ by setting their evaluations on $\mathfrak{c}$ to be equal to zero. Let η be the 1-form on $\mathfrak{g}$ such that $\eta(X) = 0$ for all X in $\mathfrak{g}$ and $\eta(F) = 1$. Next, for any X in $\mathfrak{h}$ and any α in $\mathfrak{h}^*$, we have

$$\mathcal{L}_X\eta = -\iota_X\omega, \quad \text{and} \quad \mathcal{L}_F\alpha = 0. \tag{5.2}$$

Suppose that

$$\Phi = \begin{pmatrix} \varphi & \pi^\sharp \\ \theta^\flat & -\varphi^* \end{pmatrix} \tag{5.3}$$

is a generalized complex structure on the Lie group H as given above. Suppose, in addition, that all three tensorial components φ, θ, and π are left-invariant. Then we treat Φ a real linear map from $\mathfrak{h} \oplus \mathfrak{h}^*$, and extend it by zeros to a linear map from $\mathfrak{g} \oplus \mathfrak{g}^*$. It follows that $\mathcal{J} = (F, \eta, \pi, \theta, \varphi)$ defines a generalized almost contact structure on the Lie algebra $\mathfrak{g}$, and hence a left-invariant generalized almost contact structure on the Lie group G, whose algebra is determined by (5.1).

With respect to the notation in Section 2.2 and as far as invariant sections are concerned,

$$\ker \eta = \mathfrak{h}, \quad \ker F = \mathfrak{h}^*. \tag{5.4}$$

The spaces of invariant sections of L and L^* are the finite-dimensional complex vector spaces

$$\mathfrak{l} = \langle F\rangle_{\mathbb{C}} \oplus \mathfrak{h}^{1,0}, \quad \text{and} \quad \mathfrak{l}^* = \langle \eta\rangle_{\mathbb{C}} \oplus \mathfrak{h}^{0,1}, \tag{5.5}$$

respectively. Due to the structure equations (5.1) and (5.2), we have $[\![\mathfrak{l}, \mathfrak{l}]\!] = [\![\mathfrak{h}^{1,0}, \mathfrak{h}^{1,0}]\!]$, and $[\![\mathfrak{h}^{1,0}, \mathfrak{h}^{1,0}]\!] \subseteq \langle F\rangle_{\mathbb{C}} \oplus \mathfrak{h}_{\mathbb{C}}$. Since Φ is an integrable generalized complex structure, the $\mathfrak{h}_{\mathbb{C}}$-component of $[\![\mathfrak{h}^{1,0}, \mathfrak{h}^{1,0}]\!]$ is contained in $\mathfrak{h}^{1,0}$. Therefore, $[\![\mathfrak{l}, \mathfrak{l}]\!] \subseteq \mathfrak{l}$. From (5.1), we also see that $[\![\mathfrak{l}^*, \mathfrak{l}^*]\!]$ is not, in general, a subspace of $\mathfrak{l}^*$. Therefore, we obtain an invariant generalized contact structure, but not a strong one.

5.2. *Geometry on a four-dimensional Kodaira manifold*

In [**20**], the first author showed that the complex structure on a primary Kodaira surface could be deformed, within a family of generalized complex structures, to a symplectic structure. In this section, we briefly recall his construction to establish the notation.

A real four-dimensional Kodaira manifold N is a cocompact quotient of a four-dimensional nilpotent Lie group H [**9**]. Let $\{e_1, e_2, e_3, e_4\}$ be a basis of the Lie algebra $\mathfrak{h}$, and let $\{e^1, \ldots, e^4\}$ be the dual basis. The sole non-zero structure equation and its dual expression are given by

$$[e_1, e_2] = e_3 \quad \text{and} \quad de^3 = -e^1 \wedge e^2, \tag{5.6}$$

respectively. In particular, the space of invariant closed 2-forms on the Kodaira manifold N is spanned by

$$e^1 \wedge e^3 - e^2 \wedge e^4, \quad e^1 \wedge e^4 + e^2 \wedge e^3, \quad e^1 \wedge e^3 + e^2 \wedge e^4, \quad e^1 \wedge e^4 - e^2 \wedge e^3. \tag{5.7}$$

For any real constants u_1, v_1, u_2, and v_2 with $u_1^2 + v_1^2 - u_2^2 - v_2^2 \neq 0$, we have that

$$\begin{aligned} &u_1(e^1 \wedge e^3 - e^2 \wedge e^4) + v_1(e^1 \wedge e^4 + e^2 \wedge e^3) \\ &\quad + u_2(e^1 \wedge e^3 + e^2 \wedge e^4) + v_2(e^1 \wedge e^4 - e^2 \wedge e^3) \end{aligned} \tag{5.8}$$

is a symplectic form.

On the other hand, the group H has an invariant integrable complex structure J. In terms of the given basis for the Lie algebra $\mathfrak{h}$, we have

$$Je_1 = e_2, \quad Je_2 = -e_1, \quad Je_3 = e_4, \quad Je_4 = -e_3.$$

This complex structure on H descends to an integrable complex structure on N. It turns N into a compact complex surface. In this situation, N is known as a Kodaira surface. One of the key results in [**20**] is the following.

PROPOSITION 5.1. *On the Kodaira surface N, the complex structure J and the symplectic structures*

$$u_1(e^1 \wedge e^3 - e^2 \wedge e^4) + v_1(e^1 \wedge e^4 + e^2 \wedge e^3), \quad u_1^2 + v_1^2 \neq 0, \tag{5.9}$$

are in the same deformation family of generalized complex structures.

The deformation family could be given explicitly in terms of a choice of an $(-i)$-eigenspace of an invariant generalized complex structure. Choose an ordered basis for $(\mathfrak{h}\oplus\mathfrak{h}^*)_{\mathbb{C}}$ as follows:

$$\frac{1}{2}(e_1+ie_2),\quad \frac{1}{2}(e_3+ie_4),\quad e^1+ie^2,\quad e^3+ie^4,$$
$$\frac{1}{2}(e_1-ie_2),\quad \frac{1}{2}(e_3-ie_4),\quad e^1-ie^2,\quad e^3-ie^4.$$

Then the $(-i)$-eigenspace is spanned by the following row vectors:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & t_3 & 0 & 0 & t_1 \\ 0 & 1 & 0 & 0 & 0 & t_2 & -t_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & t_4 & -t_3 & 0 \\ 0 & 0 & 0 & 1 & -t_4 & 0 & 0 & -t_2 \end{pmatrix}, \tag{5.10}$$

where $t_1,\dots,t_4$ are complex numbers. When all of them are equal to zero, the distribution is due to the classical complex structure J. When $t_1=t_4=0$, this distribution is due to a generic classical complex structure. On the other hand, the generalized complex structure determined by the symplectic form given by (5.9) is contained in this family with $t_2=t_3=0$ and

$$t_1=\frac{i}{2}(u_1+iv_1),\quad t_4=\frac{2i}{u_1-iv_1}=\frac{1}{t_1}.$$

Note that not all symplectic forms on the Kodaira manifold are contained in the family (5.10). However, due to a combination of (5.8) with (5.10), the complex structure J and all symplectic forms on N are contained in the same connected component of the generalized deformation family.

5.3. *Geometry on an* SO(2)*-bundle over a Kodaira surface*

Now we apply the general construction in Section 5.1 to the Kodaira manifold N. Choose the symplectic form

$$\omega=-(e^1\wedge e^3-e^2\wedge e^4).$$

Let M be the principal SO(2)-bundle on N with characteristic class $-\omega$. It is covered by a five-dimensional simply-connected nilpotent group G, which is a central extension of H. Let e_5 be the fundamental vector field of the principal bundle. Let e^5 be a connection 1-form. Then the structure equations on $\mathfrak{g}$ are

$$[e_1,e_2]=e_3,\quad [e_1,e_3]=-e_5,\quad [e_2,e_4]=e_5. \tag{5.11}$$

The dual structure equations in terms of the Chevalley–Eilenberg differential are

$$de^3=-e^1\wedge e^2,\quad de^5=-\omega=e^1\wedge e^3-e^2\wedge e^4. \tag{5.12}$$

Treating e^5 as a contact 1-form on G, we construct its associated generalized contact structure $\mathcal{J}_1=(F,\eta,\pi,\theta,\varphi)$ as given in Section 3.1. We have

$$F=e_5,\quad \eta=e^5,\quad \pi=e_1\wedge e_3-e_2\wedge e_4,\quad \theta=e^1\wedge e^3-e^2\wedge e^4,\quad \varphi=0.$$

On the other hand, due to the construction in Section 5.1, the complex structure J on H induces a generalized contact structure $\mathcal{J}_0$ with

$$F=e_5,\quad \eta=e^5,\quad \pi=0,\quad \theta=0,$$
$$\varphi=e_2\otimes e^1-e_1\otimes e^2+e_4\otimes e^3-e_3\otimes e^4.$$

All invariant objects on G descend to a cocompact quotient M. As a result of the general construction in Section 5.1 and Proposition 5.1, we have the following conclusion.

PROPOSITION 5.2. *The generalized contact structure $\mathcal{J}_1$ on the manifold M, determined by the contact 1-form e^5, and the generalized contact structure $\mathcal{J}_0$ are in the same deformation family of generalized contact structures.*

Finally, note that the generalized contact structure $\mathcal{J}_0$ is not strong in the sense that the space of sections of L^* is not closed with respect to the Courant bracket on the manifold M. One may check this directly through the given structure equations. One may also observe that $d\eta = -\omega = e^1 \wedge e^3 - e^2 \wedge e^4$. With respect to the given φ, it is of type $(2,0)+(0,2)$. Therefore, the obstruction for the integrability of L^* does not vanish. Due to Lemma 3.3, the triple (F, η, φ) on M is *not* a normal almost contact structure.

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References

1. V. I. ARNOLD, *Mathematical methods of classical mechanics*, Graduate Texts in Mathematics 60 (Springer, New York, 1974).
2. D. E. BLAIR, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics 203 (Birkhäuser, Boston, MA, 2001).
3. W. M. BOOTHBY and H. C. WANG, 'On contact manifolds', *Ann. of Math.* 68 (1958) 721–734.
4. C. P. BOYER and K. GALICKI, *Sasakian geometry* (Oxford University Press, Oxford, 2008).
5. S. S. CHERN, *Pseudo-groupes continus infinis* (Colloques Internationaux du CNRS, Strasbourg, 1953) 119–136.
6. S. S. CHERN, 'The geometry of G-structures', *Bull. Amer. Math. Soc.* 72 (1966) 167–219.
7. T. COURANT, 'Dirac structures', *Trans. Amer. Math. Soc.* 319 (1990) 631–661.
8. A. FUJIMOTO, *Theory of G-structures*, Publications of the Study Group of Geometry 1 (Okayama University, Okayama, 1972).
9. G. GRANTCHAROV, C. MACLAUGHLIN, H. PEDERSEN and Y. S. POON, 'Lie groups, Kodaira manifolds and complex deformations', *Glasg. Math. J.* 46 (2004) 259–281.
10. J. W. GRAY, 'Some global properties of contact structures', *Ann. of Math.* 69 (1959) 421–450.
11. M. GUALTIERI, 'Generalized complex geometry', DPhil Thesis, University of Oxford, 2003, arXiv: math.DG/0401221.
12. V. GUILLEMIN and S. STERNBERG, *Symplectic techniques in physics* (Cambridge University Press, Cambridge, 1990).
13. N. J. HITCHIN, 'Generalized Calabi–Yau manifolds', *Q. J. Math.* 54 (2003) 281–308.
14. D. IGLESIAS-PONTE and A. WADE, 'Contact manifolds and generalized complex structures', *J. Geom. Phys.* 53 (2005) 249–258.
15. P. LIBERMANN, 'Sur les automorphismes infinitésimaux des structures symplectiques et de structures de contact', *Colloque de géometrie differentielle globale*, Brussels (Gauthier-Villars, Paris, 1958) 37–59.
16. S. LIE, *Theorie der transformationsgruppen*, Abschnitt 2, (Teubner, Leipzig, 1890).
17. Z. J. LIU, A. WEINSTEIN and P. XU, 'Manin triples for Lie bialgebroids', *J. Differential Geom.* 45 (1997) 547–574.
18. K. MACKENZIE, *General theory of Lie groupoids and Lie algebroids*, London Mathematical Society Lecture Note Series 213 (Cambridge University Press, Cambridge, 2005).
19. J. MOSER, 'On the volume elements on a manifold', *Trans. Amer. Math. Soc.* 120 (1965) 286–294.
20. Y. S. POON, 'Extended deformation of Kodaira surfaces', *J. reine angew. Math.* 590 (2006) 45–65.
21. Y. S. POON and A. WADE, 'Approaches to generalize contact structures', *Q. J. Pure Appl. Math.* 6 (2010) 603–622.
22. D. ROYTENBERG, 'Quasi-Lie bialgebroids and twisted Poisson manifolds', *Lett. Math. Phys.* 61 (2002) 123–137.
23. S. SASAKI, 'On differentiable manifolds with certain structures which are closely related to almost contact structure I', *Tôhoku Math. J.* 12 (1960) 459–476.

24. I. Vaisman, 'Dirac structures and generalized complex structures on $TM \times \mathbb{R}^h$, *Adv. Geom.* 7 (2007) 453–474.
25. I. Vaisman, 'Generalized CRF-structures', *Geom. Dedicata* 133 (2008) 129–154.

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