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Holomorphic Poisson Cohomology

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Abstract: Holomorphic Poisson structures arise naturally in the realm of generalized geometry. A holomorphic Poisson structure induces a deformation of the complex structure in a generalized sense, whose cohomology is obtained by twisting the Dolbeault $\bar{\partial}$ -operator by the holomorphic Poisson bivector field. Therefore, the cohomology space naturally appears as the limit of a spectral sequence of a double complex. The first sheet of this spectral sequence is simply the Dolbeault cohomology with coefficients in the exterior algebra of the holomorphic tangent bundle. We identify various necessary conditions on compact complex manifolds on which this spectral sequence degenerates on the level of the second sheet. The manifolds to our concern include all compact complex surfaces, Kähler manifolds, and nilmanifolds with abelian complex structures or parallelizable complex structures.

1 Introduction

The algebraic geometry of Poisson brackets over complex manifolds were studied for some time [22]. In recent years, there are significant interests on holomorphic Poisson structures due to their emergence in generalized complex geometry [12] [13] [14]. As such, one could consider their deformations either as complex analytic objects [7] [9] [15], or as generalized complex objects [10]. On the other hand, there are classifications of holomorphic Poisson structures on algebraic surfaces [2], and computations of their related cohomology theory [17].

A holomorphic Poisson structure consists of a holomorphic bi-vector field Λ on a complex manifold such that $[\Lambda, \Lambda] = 0$. It is demonstrated in [10] that the deformation of such structure as a generalized complex structure is dictated by the cohomology space $\oplus_k H_\Lambda^k$ of the differential operator $\bar{\partial}_\Lambda = \bar{\partial} + [\Lambda, -]$. This observation motivates the authors' desire to find a way to compute the cohomology spaces H_Λ^k . Since this space is naturally the limit of a spectral sequence of a double complex due to the operators $\bar{\partial}$ and $[\Lambda, -]$, it is an obvious question on when and how fast this spectral sequence could degenerate. We will see that the first sheet of this spectral sequence is the Dolbeault cohomology on the manifold with coefficients in the holomorphic polyvectors. As this is a classical object, it becomes desirable to identify the conditions under which the spectral sequence will degenerate at its second level.

In this paper, we identify three situations in which the spectral sequence of the holomorphic Poisson double complex degenerates at its second level. In the next chapter, we will explain the role of holomorphic Poisson structures in generalized geometry, and then set up our computation of cohomology in the context of Lie bi-algebroids and their dual differentials [18]. Our reference for differential calculus of Lie algebroids is the book [19]. We will then set up the holomorphic Poisson double complex and its related spectral sequence.

We will make observation on three very different kinds of complex manifolds: complex parallelizable, nilmanifolds with abelian complex structures, and Kählerian manifolds.

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On complex parallelizable manifolds, we have the following result.

Theorem A. *Let M be a complex parallelizable manifold with an invariant holomorphic Poisson structure Λ . Let $\mathfrak{g}^{1,0}$ be the vector space of holomorphic vector fields on M . Let $H_{\delta}^q(\mathfrak{g}^{*(0,1)})$ be the Chevalley-Eilenberg cohomology for the conjugation algebra $\mathfrak{g}^{0,1}$, and $H_{\text{ad}_\Lambda}^p(\mathfrak{g}^{1,0})$ the cohomology of the operator $\text{ad}_\Lambda = \llbracket \Lambda, - \rrbracket$ on the exterior algebra of $\mathfrak{g}^{1,0}$, then*

$$H_{\Lambda}^k(M) = \bigoplus_{p+q=k} H_{\delta}^q(\mathfrak{g}^{*(0,1)}) \otimes H_{\text{ad}_\Lambda}^p(\mathfrak{g}^{1,0}).$$

This result will appear as Theorem 3 in this text. The proof of this result is not hard except for an application of Sakane's work [26] on Dolbeault cohomology on complex parallelizable manifolds because the holomorphic tangent bundle is trivial in this case. A typical example is the complex three-dimensional Iwasawa manifold.

The analysis of nilmanifolds takes significantly more effort. Much are based on our experience with computation of various cohomology on nilmanifolds, starting from Nomizu's work on de Rham cohomology [21], its generalization to Dolbeault cohomology [3] [4] [6] and their generalizations to cohomology with coefficients in holomorphic vector fields [10] [11] [20]. In this paper, we need to deal with cohomology with coefficients in holomorphic poly-vector fields. The result below generalizes what was known for Kodaira surfaces [23].

Theorem B. *Suppose that $M = G/\Gamma$ is a 2-step nilmanifold with an abelian complex structure. Suppose that the center of \mathfrak{g} is real two-dimensional, then for any invariant holomorphic Poisson structure Λ , the Poisson spectral sequence degenerates on the second sheet.*

This theorem will appear as Theorem 2. In view of the work in [5] and [25], we expect much room to generalize this theorem. There are also lots of examples.

Finally, to deal with Kählerian manifolds, we adopt an idea from Hitchin in his paper [15] (see also [9]) when he uses a contraction of the holomorphic bivector field Λ to pass a problem at hand to a consideration on Dolbeault cohomology. This contraction yields a homomorphism of double complex. Based on it, we prove the following result.

Theorem C. *Suppose that X is a holomorphic Poisson manifold with complex dimension n . If the Frölicher spectral sequence of its complex structure degenerates at the E_1 -level, then its holomorphic Poisson spectral sequence degenerates at the E_n -level.*

This theorem will appear as Theorem 6. Note that it is well known that the Frölicher spectral sequence on a compact complex surface always degenerates [1]. The last theorem could be applied to all compact complex surfaces as well as all compact Kählerian manifolds [28].

2 Holomorphic Poisson Double Complex

2.1 Holomorphic Poisson Structures as Generalized Complex Structures

A generalized complex manifold [12] [13] is a smooth $2n$ -dimensional manifold M equipped with a subbundle L of the bundle $\mathcal{T} = (TM \oplus T^*M)_{\mathbb{C}}$ such that

- L and its conjugate bundle \bar{L} are transversal;
- L is maximally isotropic with respect to the natural pairing on $(TM \oplus T^*M)_{\mathbb{C}}$;
- the space of sections of L is closed with respect to the Courant bracket.

Equivalently, it is determined by a bundle automorphism \mathcal{J} on $TM \oplus T^*M$ such that $\mathcal{J} \circ \mathcal{J} = -\text{identity}$, \mathcal{J} is orthogonal with respect to the natural pairing, and the space of sections of the $+i$ -eigenbundle with respect to \mathcal{J} is closed with respect to the Courant bracket.

Typical examples of generalized complex structures are classical complex structures and symplectic structures on a manifold. For a classical complex structure, the corresponding bundle L as a generalized complex structure is $T^{1,0}M \oplus T^{*(0,1)}M$, the direct sum of the bundle of type (1,0)-vectors and the bundle of type (0,1)-forms.

Suppose that a manifold M is equipped with a complex structure J . A holomorphic Poisson structure on (M, J) is a holomorphic bi-vector field Λ such that with respect to the Schouten bracket, $\llbracket \Lambda, \Lambda \rrbracket = 0$. It follows that for any number t , the map

$$\mathcal{J}_{t\Lambda} = \begin{pmatrix} J & \text{Im}(t\Lambda) \\ 0 & -J^* \end{pmatrix} \tag{1}$$

determines a generalized complex structure. Here by $J^* \alpha$ where α is a 1-form, we mean $(J^* \alpha)X = \alpha(JX)$ for any vector field X .

At $t = 0$, one recovers the generalized complex structure determined by the classical complex structure J on the manifold M . This is an example of deformation of generalized complex structures.

The corresponding bundles for the generalized complex structures $\mathcal{J}_{t\Lambda}$ and $-\mathcal{J}_{t\Lambda}$ are the graphs $L_{\overline{L_{t\Lambda}}}$ and $\overline{L_{t\Lambda}}$ where

$$\overline{L_{t\Lambda}} = \{ \overline{\ell} + t\Lambda(\overline{\ell}) : \overline{\ell} \in \overline{L} \}. \tag{2}$$

2.2 Holomorphic Poisson Cohomology

Given a generalized complex structure, the pair of bundles L and \overline{L} makes a Lie bialgebroid. Via the canonical non-degenerate pairing on the bundle \mathcal{T} , the bundle \overline{L} is complex linearly identified to the dual of L . Therefore, the Lie algebroid differential of \overline{L} acts on L . It extends to a differential on the exterior algebra of L . For the calculus of Lie bialgebroids, we follow the conventions in [19]. In particular, let ρ be the anchor map on the Lie algebroid L . For any element Γ in $C^\infty(M, \wedge^k L)$ and elements a_1, \dots, a_{k+1} in $C^\infty(M, \overline{L})$, the differential of Γ is

$$\begin{aligned} (\overline{\partial}_L \Gamma)(a_1, \dots, a_{k+1}) &= \sum_{r=1}^{k+1} (-1)^{r+1} \rho(a_r) \Gamma(a_1, \dots, \hat{a}_r, \dots, a_{k+1}) \\ &+ \sum_{r < s} (-1)^{r+s} \Gamma(\llbracket a_r, a_s \rrbracket, a_1, \dots, \hat{a}_r, \dots, \hat{a}_s, \dots, a_{k+1}). \end{aligned}$$

For example, if L comes from a complex structure, $L = T^{1,0}M \oplus T^{*(0,1)}M$. The differential is

$$\overline{\partial} : C^\infty(T^{1,0}M \oplus T^{*(0,1)}M) \rightarrow C^\infty((T^{1,0}M \otimes T^{*(0,1)}M) \oplus T^{*(0,2)}M). \tag{3}$$

Obviously, when $\overline{\omega}$ is a (0, 1)-form, then $\overline{\partial}\overline{\omega}$ is the classical $\overline{\partial}$ -operator in Dolbeault theory. When X is a (1, 0)-vector field, \overline{Y} a (0, 1)-vector field and ω a (1, 0)-form, then

$$(\overline{\partial}X)(\omega, \overline{Y}) = -\overline{Y}\omega(X) - X(\llbracket \omega, \overline{Y} \rrbracket) = -\overline{Y}\omega(X) + (\mathcal{L}_{\overline{Y}}\omega)(X) = \omega(\llbracket X, \overline{Y} \rrbracket). \tag{4}$$

Therefore, $\overline{\partial}X$ is identical to the so-called Riemann-Cauchy operator as in [8]. For future reference, we note the following observation, which could be proved by treating $\overline{\partial}$ as a Lie algebroid differential with respect to the Lie algebroid \overline{L} as we did in the calculus in equation (4) above.

Lemma 1. For any (2, 0)-field Λ , (1, 0)-forms ω, γ and (0, 1)-vector field \overline{Z} ,

$$(\overline{\partial}\Lambda)(\omega, \gamma, \overline{Z}) = -\overline{Z}(\Lambda(\omega, \gamma)) - \omega(\llbracket \overline{Z}, \Lambda\gamma \rrbracket) + \gamma(\llbracket \overline{Z}, \Lambda\omega \rrbracket).$$

2.3 Poisson Double Complex

We have seen that a holomorphic Poisson structure Λ could be treated as a deformation of the complex structure J . While the pair of bundles $L_{\bar{\Lambda}}$ and \bar{L}_{Λ} form a Lie bi-algebroid, so does the pair L and \bar{L}_{Λ} [18].

Proposition 1. [10] *The Lie algebroid differential of \bar{L}_{Λ} acting on L is given by*

$$\bar{\partial}_{\Lambda} = \bar{\partial} + \text{ad}_{\Lambda} : C^{\infty}(T^{1,0}M \oplus T^{*(0,1)}M) \rightarrow C^{\infty}(\wedge^2(T^{1,0}M \oplus T^{*(0,1)}M)), \tag{5}$$

where by definition $\text{ad}_{\Lambda}(v + \bar{w}) = \llbracket \Lambda, v + \bar{w} \rrbracket$ when v is a smooth section of $T^{1,0}M$ and \bar{w} is a smooth section of $T^{*(0,1)}M$.

The operator $\bar{\partial}_{\Lambda}$ extends to act on the exterior algebra of $T^{1,0}M \oplus T^{*(0,1)}M$:

$$\bar{\partial}_{\Lambda} : K^n \rightarrow K^{n+1}, \quad \text{where } K^n = C^{\infty}(\wedge^n(T^{1,0}M \oplus T^{*(0,1)}M)). \tag{6}$$

Since Λ is a holomorphic Poisson structure, $\bar{\partial}_{\Lambda} \circ \bar{\partial}_{\Lambda} = 0$. Therefore, one has a complex with $\bar{\partial}_{\Lambda}$ being a differential. Starting with $k = 0$, one has the k -th cohomology of this complex. It is denoted by $H_{\Lambda}^k(M)$. This cohomology space is called the k -th holomorphic Poisson cohomology of Λ ; or simply Poisson cohomology.

Since Λ is holomorphic Poisson, it follows that

$$\bar{\partial} \circ \bar{\partial} = 0, \quad \bar{\partial} \circ \text{ad}_{\Lambda} + \text{ad}_{\Lambda} \circ \bar{\partial} = 0, \quad \text{ad}_{\Lambda} \circ \text{ad}_{\Lambda} = 0. \tag{7}$$

Define $A^{p,q} = C^{\infty}(M, \wedge^p T^{1,0}M \otimes \wedge^q T^{*(0,1)}M)$, then

$$\bar{\partial} : A^{p,q} \rightarrow A^{p,q+1}; \quad \text{and} \quad \text{ad}_{\Lambda} : A^{p,q} \rightarrow A^{p+1,q}. \tag{8}$$

Therefore, we obtain a double complex.

$$\begin{array}{ccc} A^{p,q+1} & \longrightarrow & A^{p+1,q+1} \\ \bar{\partial} \uparrow & & \uparrow \\ A^{p,q} & \longrightarrow & A^{p+1,q} \\ & & \text{ad}_{\Lambda} \end{array} .$$

For the double complex $(A^{*,*}, \text{ad}_{\Lambda}, \bar{\partial})$, its associated single complex is $K^n = \oplus_{p+q=n} A^{p,q}$ and the differential is

$$\bar{\partial}_{\Lambda} = \bar{\partial} + \text{ad}_{\Lambda} : K^n \longrightarrow K^{n+1}.$$

Lemma 2. *The triples $(A^{*,*}, \text{ad}_{\Lambda}, \bar{\partial})$ form a double complex. Its total cohomology is the holomorphic Poisson cohomology.*

The bi-grading yields two spectral sequences abutting to the Poisson cohomology. We focus on the case when the filtration is given by

$$F^p K^n = \oplus_{p'+q=n, p' \geq p} A^{p',q}. \tag{9}$$

Equivalently, the zero-th sheet of this spectral sequence $E_0^{p,q}$ is precisely $A^{p,q}$. Since $\bar{\partial}_{\Lambda} = \bar{\partial} + \text{ad}_{\Lambda}$ and ad_{Λ} does not raise the second index in the bi-degree (p, q) , the restriction of the differential on the 0-th sheet is

$$\bar{\partial} : A^{p,q} \rightarrow A^{p,q+1}. \tag{10}$$

Therefore, the first sheet of this spectral sequence is the Dolbeault cohomology

$$E_1^{p,q} = H^q(M, \Theta^p), \tag{11}$$

where Θ^p is the p -th exterior power of the holomorphic tangent bundle of the complex manifold M .

It follows that the next sheet of this spectral sequence is given by the cohomology of the differential

$$d_1^{p,q} = \text{ad}_\Lambda : H^q(M, \Theta^p) \rightarrow H^q(M, \Theta^{p+1}) \tag{12}$$

for all $p + q \geq 0$, and the second sheet of this spectral sequence is

$$E_2^{p,q} = \frac{\text{kernel of } \text{ad}_\Lambda : H^q(M, \Theta^p) \rightarrow H^q(M, \Theta^{p+1})}{\text{image of } \text{ad}_\Lambda : H^q(M, \Theta^{p-1}) \rightarrow H^q(M, \Theta^p)}. \tag{13}$$

To facilitate further computation, we recall that the differential $d_2^{p,q}$ is computed as follows. Suppose that ‘ v ’ represents a class in $H^q(M, \Theta^p)$ and $\text{ad}_\Lambda v$ represents a zero class in $H^q(M, \Theta^{p+1})$, it represents an element in $E_2^{p,q}$. As $\text{ad}_\Lambda v$ represents a zero class, there exists ‘ w ’ a section of $T^{p+1,0}M \otimes T^{*(0,q-1)}M$ such that $\bar{\partial}w = \text{ad}_\Lambda v$. Since

$$\bar{\partial}\text{ad}_\Lambda w = -\text{ad}_\Lambda \bar{\partial}w = -\text{ad}_\Lambda \text{ad}_\Lambda v = 0,$$

$\text{ad}_\Lambda w$ represents a class in $E_2^{p+2,q-1}$. By definition, it is the image of the class of v via the map $d_2^{p,q}$.

2.4 Complex Projective Spaces as Examples

Let Λ be any non-trivial holomorphic section of the bundle Θ^2 of the complex projective space $M = \mathbb{C}\mathbb{P}^n$. Assume that $\llbracket \Lambda, \Lambda \rrbracket = 0$ so that Λ is a holomorphic Poisson structure. Since $H^q(M, \Theta^p)$ vanishes for all $q \geq 1$, the first sheet of the holomorphic Poisson double complex is reduced to its zero-th row:

$$H^0(M, \mathcal{O}) \xrightarrow{d_1^{0,0}=\text{ad}_\Lambda} H^0(M, \Theta) \xrightarrow{d_1^{1,0}=\text{ad}_\Lambda} H^0(M, \Theta^2) \rightarrow \dots$$

It is apparent that d_1 is not identically equal to zero. It is also apparent that d_2 is identically equal to zero.

2.5 Hopf Manifolds $M = S^1 \times S^{2n-1}$

Consider \mathbb{C}^n with coordinates $z = (z_1, \dots, z_n)$. Let $\lambda > 1$ be a real number. It generates a one-parameter group of automorphisms on \mathbb{C}^n . The quotient of $\mathbb{C}^n \setminus \{0\}$ with respect this group is diffeomorphic to the manifold $M = S^1 \times S^{2n-1}$. The complex structure on \mathbb{C}^n descends onto M to define an integrable complex structure. As a complex manifold, it is a principal elliptic fibration over the complex projective space $N = \mathbb{C}\mathbb{P}^{n-1}$:

$$\pi : M = S^1 \times S^{2n-1} \rightarrow N = \mathbb{C}\mathbb{P}^{n-1}.$$

Denote the fiber by F and the holomorphic vertical vector field by V . We have the following exact sequence of holomorphic vector bundles over the Hopf manifold M :

$$0 \rightarrow \mathcal{O}_X(V) \rightarrow \Theta_M \rightarrow \pi^* \Theta_N \rightarrow 0, \tag{14}$$

where $\mathcal{O}_X(V)$ denote the vertical bundle of the projection π . It is also a bundle trivialized by the vector field V . Explicitly the vector field V is simply the holomorphic vector field generated by the group of dilations.

$$V = z_1 \frac{\partial}{\partial z_1} + \dots + z_n \frac{\partial}{\partial z_n}. \tag{15}$$

For all $1 \leq p \leq n - 1$, the exact sequence (14) generates

$$0 \rightarrow \mathcal{O}_X(V) \otimes \pi^* \Theta_N^{p-1} \rightarrow \Theta_M^p \rightarrow \pi^* \Theta_N^p \rightarrow 0. \tag{16}$$

We also have $\mathcal{O}_X(V) \otimes \pi^* \Theta_N^{n-1} \cong \Theta_M^n$. Let $\bar{w} = \bar{\partial} \ln |z|^2$. It is a global $\bar{\partial}$ -closed $(0, 1)$ -form on the Hopf manifold M .

Proposition 2. *On the Hopf manifold M ,*

- for all $p \geq 0$ and $q \geq 2$, $H^q(M, \Theta_M^p) = 0$.
- $H^0(M, \mathcal{O}_M) = \langle 1 \rangle$, and $H^1(M, \mathcal{O}_M) = \langle \bar{\omega} \rangle$.
- $H^0(M, \Theta_M^n) = \langle V \rangle \otimes H^0(N, \Theta_N^{n-1})$, $H^1(M, \Theta_M^n) = \langle \bar{\omega} \wedge V \rangle \otimes H^0(N, \Theta_N^{n-1})$.
- For all $1 \leq p \leq n - 1$,

$$\begin{aligned} H^0(M, \Theta_M^p) &= H^0(N, \Theta_N^p) \oplus \langle V \rangle \otimes H^0(N, \Theta_N^{p-1}), \\ H^1(M, \Theta_M^p) &= \langle \bar{\omega} \rangle \otimes H^0(M, \Theta_M^p) \\ &= \langle \bar{\omega} \rangle \otimes H^0(N, \Theta_N^p) \oplus \langle \bar{\omega} \wedge V \rangle \otimes H^0(N, \Theta_N^{p-1}). \end{aligned}$$

The proof of this proposition is a straightforward application of the Leray spectral sequence with respect to the fibration π . Similar computation could be found in [11] and [16]. Below we outline a proof. First of all, by virtue of the dimension of the fiber and the global (0, 1)-form $\bar{\omega}$ on the manifold M , one proves that the direct image sheaves of $R^q \pi_* \mathcal{O}_M = 0$ for $q \geq 2$, and

$$R^0 \pi_* \mathcal{O}_M \cong \mathcal{O}_N, \quad R^1 \pi_* \mathcal{O}_M \cong \mathcal{O}_N(\bar{\omega}).$$

By the Projective Formula, for all $k \geq 0$, $R^q \pi_* \pi^* \Theta_N^k = 0$ for $q \geq 2$, and

$$R^0 \pi_* \pi^* \Theta_N^k \cong \Theta_N^k, \quad R^1 \pi_* \pi^* \Theta_N^k \cong \mathcal{O}_N(\bar{\omega}) \otimes \Theta_N^k.$$

As the second sheet of the Leray spectral sequence for $\pi^* \Theta_N^k$ is

$$E_2^{p,q}(\pi^* \Theta_N^k) = H^p(N, R^q \pi_* \pi^* \Theta_N^k).$$

$E_2^{p,q} = 0$ for all $q \geq 2$. Moreover,

$$E_2^{p,0}(\pi^* \Theta_N^k) = H^p(N, \Theta_N^k), \quad E_2^{p,1}(\pi^* \Theta_N^k) = \langle \bar{\omega} \rangle \otimes H^p(N, \Theta_N^k).$$

Recall that N is the complex projective space $\mathbb{C}P^{n-1}$. Therefore, the above cohomology spaces vanish except when $p = 0$. It follows immediately the differential $d_2^{p,q}$ is identically zero, and hence the Leray spectral sequence degenerates to $\bigoplus_{p+q=m} E_2^{p,q}(\pi^* \Theta_N^k) = H^m(M, \pi^* \Theta_N^k)$. In particular,

$$H^0(M, \pi^* \Theta_N^k) \cong H^0(N, \Theta_N^k), \quad H^1(M, \pi^* \Theta_N^k) \cong \langle \bar{\omega} \rangle \otimes H^0(N, \Theta_N^k). \tag{17}$$

By the exact sequence (16), one concludes immediately, that for all $p \geq 0$ and $q \geq 2$, $H^q(M, \Theta_M^p) = 0$. By chasing diagram, one could actually show that the induced long exact sequence of (16) splits. The claim in the proposition above follows.

Given the proposition above, one notes that the non-zero terms of the first sheet of the Poisson double complex with respect to any holomorphic Poisson structure on the Hopf manifold are contained in the zeroth and the first rows only. To compute the differential on the second sheet explicitly, note that the first sheet is given by

$$\begin{array}{ccccccc} \langle \bar{\omega} \rangle & \xrightarrow{ad_\Lambda} & \langle \bar{\omega} \rangle \otimes H^0(M, \Theta_M) & \xrightarrow{ad_\Lambda} & \langle \bar{\omega} \rangle \otimes H^0(M, \Theta_M^2) & \xrightarrow{ad_\Lambda} & \langle \bar{\omega} \rangle \otimes H^0(M, \Theta_M^3) & \xrightarrow{ad_\Lambda} & \dots \\ \langle 1 \rangle & \xrightarrow{ad_\Lambda} & H^0(M, \Theta_M) & \xrightarrow{ad_\Lambda} & H^0(M, \Theta_M^2) & \xrightarrow{ad_\Lambda} & H^0(M, \Theta_M^3) & \xrightarrow{ad_\Lambda} & \dots \end{array}$$

To prepare our computation, note that when we treat $\mathbb{C}P^{n-1}$ as a homogeneous space, then its space of holomorphic vector fields is $\mathfrak{sl}(n, \mathbb{C})$. They are generated by linear action of $SU(n)$ on \mathbb{C}^n .

Let $A \in \mathfrak{sl}(n, \mathbb{C})$, and define on \mathbb{C}^n the function

$$f_A(z) = \frac{1}{|z|^2} \bar{z}^T A z.$$

By homogeneity, this function is well-defined on the Hopf manifold M . It is straightforward to show that $\mathcal{L}_A \bar{\omega} = \bar{\partial} f_A$ and $\mathcal{L}_V f_A = 0$.

Consider the bi-vector field $\Lambda = V \wedge A$. Since A and V commute, Λ is a holomorphic Poisson structure. We claim that its associated Poisson double complex degenerates at the E_2 -level.

Since the only non-trivial elements in the first sheet of the double complex are in the zeroth and the first row, $E_2^{p,q}$ vanishes except possibly when $(p, q) = (p, 0)$ and $(p, q) = (p, 1)$. Therefore, the only possibly non-trivial differential on the second sheet are $d_2^{p,1}$. According to the structure of the cohomology spaces, each term in $E_2^{p,1}$ is represented by an element of the form $\bar{\omega} \wedge \Gamma$ for some Γ in $H^0(M, \theta^p)$. Since $\mathcal{L}_V(\bar{\omega}) = 0$ and $\mathcal{L}_V(\Gamma) = 0$,

$$\begin{aligned} \llbracket V \wedge A, \bar{\omega} \wedge \Gamma \rrbracket &= V \wedge \llbracket A, \bar{\omega} \wedge \Gamma \rrbracket = V \wedge \mathcal{L}_A \bar{\omega} \wedge \Gamma + (-1)^p V \wedge \llbracket A, \Gamma \rrbracket \wedge \bar{\omega} \\ &= V \wedge \bar{\partial} f_A \wedge \Gamma + (-1)^p V \wedge \llbracket A, \Gamma \rrbracket \wedge \bar{\omega} = -\bar{\partial}(f_A V \wedge \Gamma) + (-1)^p V \wedge \llbracket A, \Gamma \rrbracket \wedge \bar{\omega}. \end{aligned}$$

So, $\text{ad}_\Lambda(\bar{\omega} \wedge \Gamma)$ represents the zero class in $\langle \bar{\omega} \rangle \otimes H^0(M, \theta^{p+1})$ if and only if $\llbracket A, \Gamma \rrbracket$ is equal to zero. Under this condition, $\text{ad}_\Lambda(\bar{\omega} \wedge \Gamma) = -\bar{\partial}(f_A V \wedge \Gamma)$, and $d_2^{p,1}(\bar{\omega} \wedge \Gamma)$ is represented by

$$\begin{aligned} -\text{ad}_\Lambda(f_A V \wedge \Gamma) &= -\llbracket V \wedge A, f_A V \wedge \Gamma \rrbracket = A \wedge \llbracket V, f_A V \wedge \Gamma \rrbracket - V \wedge \llbracket A, f_A V \wedge \Gamma \rrbracket \\ &= A \wedge (\mathcal{L}_V f_A) V \wedge \Gamma + A \wedge f_A \llbracket V, V \wedge \Gamma \rrbracket - V \wedge (\mathcal{L}_A f_A) V \wedge \Gamma - V \wedge f_A \llbracket A, V \wedge \Gamma \rrbracket. \end{aligned}$$

Note that the first summand is equal to zero because $\mathcal{L}_V f_A = 0$. The second summand is equal to zero because V commutes with every element in $H^0(M, \theta^p)$. The third summand is equal to zero by skew-symmetry of exterior multiplication. The last summand is equal to zero because as noted above when the term $\llbracket \text{ad}_\Lambda, \bar{\omega} \wedge \Gamma \rrbracket$ represents the zero class in $H^1(M, \theta^{p+1})$, $\llbracket A, \Gamma \rrbracket = 0$. Therefore, $d_2^{p,1} = 0$ for all p .

In the rest of this article, we investigate the degeneracy of the holomorphic Poisson spectral sequence in various special settings.

3 Nilmanifolds

A compact manifold M is called a nilmanifold if there exists a simply-connected nilpotent Lie group G and a lattice subgroup Γ such that M is diffeomorphic to G/Γ . We denote the Lie algebra of G by \mathfrak{g} , and the center of \mathfrak{g} by $Z(\mathfrak{g})$. The step of the nilmanifold is the nilpotence of the Lie algebra \mathfrak{g} . A left-invariant complex structure J on G is said to be abelian if on the Lie algebra \mathfrak{g} , it satisfies the conditions $J \circ J = -\text{identity}$ and $\llbracket JA, JB \rrbracket = \llbracket A, B \rrbracket$ for all A and B in the Lie algebra \mathfrak{g} . If one complexifies the algebra \mathfrak{g} and denotes the $+i$ and $-i$ eigenspaces of J respectively by $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$, then the invariant complex structure J being abelian is equivalent to the complex algebra $\mathfrak{g}^{1,0}$ being abelian.

Denote $\wedge^k \mathfrak{g}^{1,0}$ and $\wedge^k \mathfrak{g}^{*(0,1)}$ respectively by $\mathfrak{g}^{k,0}$ and $\mathfrak{g}^{*(0,k)}$.

3.1 Cohomology of Nilmanifolds

Now assume that the Lie algebra \mathfrak{g} is 2-step nilpotent, i.e. $[\mathfrak{g}, \mathfrak{g}] \subset Z(\mathfrak{g})$. In such a case, we call $M = G/\Gamma$ a 2-step nilmanifold [20].

On the nilmanifold M , we consider $\mathfrak{g}^{k,0}$ as invariant $(k, 0)$ -vector fields, and $\mathfrak{g}^{*(0,k)}$ as invariant $(0, k)$ -forms. It yields an inclusion map

$$\mathfrak{g}^{p,0} \otimes \mathfrak{g}^{*(0,q)} \hookrightarrow C^\infty(M, \wedge^p T^{1,0} M \otimes \wedge^q T^{*(0,1)} M).$$

When the complex structure is also invariant, $\bar{\partial}$ sends $\mathfrak{g}^{p,0} \otimes \mathfrak{g}^{*(0,q)}$ to $\mathfrak{g}^{p,0} \otimes \mathfrak{g}^{*(0,q+1)}$. The concerned cohomology is

$$H^q(\mathfrak{g}^{p,0}) = \frac{\text{kernel of } \bar{\partial} : \mathfrak{g}^{*(0,q)} \otimes \mathfrak{g}^{p,0} \rightarrow \mathfrak{g}^{*(0,q+1)} \otimes \mathfrak{g}^{p,0}}{\text{image of } \bar{\partial} : \mathfrak{g}^{*(0,q-1)} \otimes \mathfrak{g}^{p,0} \rightarrow \mathfrak{g}^{*(0,q)} \otimes \mathfrak{g}^{p,0}}. \tag{18}$$

The inclusion map yields a homomorphism of cohomology:

$$H^q(\mathfrak{g}^{p,0}) \hookrightarrow H^q(M, \Theta^p).$$

Theorem 1. *On a 2-step nilmanifold M with an invariant abelian complex structure, the inclusion $\mathfrak{g}^{p,0} \otimes \mathfrak{g}^{*(0,q)}$ in $C^\infty(M, \wedge^p T^{1,0}M \otimes \wedge^q T^{*(0,1)}M)$ induces an isomorphism of cohomology. In other words,*

$$H^q(\mathfrak{g}^{p,0}) \cong H^q(M, \Theta^p).$$

We need some preparations before we give a proof for this theorem.

Let C be the center of G and $\psi : G \rightarrow G/C$ the quotient map. Since G is 2-step nilpotent, G/C is abelian. Consider $M = G/\Gamma$ and $N = \psi(G)/\psi(\Gamma)$. We have a holomorphic fibration $\Psi : M \rightarrow N$ whose fiber is isomorphic to $F = C/(C \cap \Gamma)$.

Let $\mathfrak{c} = Z(\mathfrak{g})$, $\mathfrak{t} = \mathfrak{g}/Z(\mathfrak{g})$. Below are some facts shown in Sections 2 and 3 of [20]. Since \mathfrak{g} is 2-step nilpotent, \mathfrak{t} is abelian. As a vector space, $\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \mathfrak{c}^{1,0}$, and $\mathfrak{g}^{*(1,0)} = \mathfrak{t}^{*(1,0)} \oplus \mathfrak{c}^{*(1,0)}$. The only nontrivial Lie brackets in $\mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ are of the form $[\mathfrak{t}^{1,0}, \mathfrak{t}^{0,1}] \subset \mathfrak{c}^{1,0} \oplus \mathfrak{c}^{0,1}$.

Explicitly, there exists a real basis $\{X_k, JX_k : 1 \leq k \leq n\}$ for \mathfrak{t} and $\{Z_\ell, JZ_\ell : 1 \leq \ell \leq m\}$ a real basis for \mathfrak{c} . The corresponding complex bases for $\mathfrak{t}^{1,0}$ and $\mathfrak{c}^{1,0}$ are respectively composed of the following elements:

$$T_k = \frac{1}{2}(X_k - iJX_k), \quad \text{and} \quad W_\ell = \frac{1}{2}(Z_\ell - iJZ_\ell). \tag{19}$$

The structure equations of \mathfrak{g} are determined by

$$[\overline{T}_k, T_j] = \sum_{\ell} E_{kj}^{\ell} W_{\ell} - \sum_{\ell} \overline{E}_{jk}^{\ell} \overline{W}_{\ell} \tag{20}$$

for some constants E_{kj}^{ℓ} . Let $\{\omega^k : 1 \leq k \leq n\}$ be the dual basis for $\mathfrak{t}^{*(1,0)}$, and let $\{\rho^{\ell} : 1 \leq \ell \leq m\}$ be the dual basis for $\mathfrak{c}^{*(1,0)}$. By formula (4),

$$\overline{\partial} T_j = \sum_{k,\ell} E_{kj}^{\ell} \overline{\omega}^k \wedge W_{\ell}, \tag{21}$$

the dual structure equations for (20) are

$$d\rho^{\ell} = \sum_{i,j} E_{ji}^{\ell} \omega^i \wedge \overline{\omega}^j. \tag{22}$$

Equivalently,

$$d\overline{\rho}^{\ell} = -\sum_{i,j} \overline{E}_{ji}^{\ell} \omega^j \wedge \overline{\omega}^i. \tag{23}$$

It follows that

$$[T_j, \overline{\rho}^{\ell}] = \mathcal{L}_{T_j} \overline{\rho}^{\ell} = -\sum_i \overline{E}_{ji}^{\ell} \overline{\omega}^i. \tag{24}$$

In summary, the operator $d = \partial + \overline{\partial}$ satisfies

$$\partial \mathfrak{t}^{*(1,0)} = 0, \quad \overline{\partial} \mathfrak{t}^{*(1,0)} = 0, \quad \partial \mathfrak{c}^{*(1,0)} = 0, \quad \overline{\partial} \mathfrak{c}^{*(1,0)} \subset \mathfrak{t}^{*(1,1)}. \tag{25}$$

Moreover, the co-boundary operator $\overline{\partial} : \mathfrak{g}^{1,0} \rightarrow \mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0}$ satisfies

$$\overline{\partial} \mathfrak{c}^{1,0} = 0, \quad \overline{\partial} \mathfrak{t}^{1,0} \subset \mathfrak{t}^{*(0,1)} \otimes \mathfrak{c}^{1,0}. \tag{26}$$

Lemma 3. *Let Θ_N be the tangent sheaf of N , the quotient manifold. Then $R^q \Psi_*(\wedge^l \Psi^* \Theta_N) = \mathfrak{c}^{*(0,q)} \otimes \wedge^l \Theta_N$.*

Proof. By Lemma 4 in [20], we have $R^q \Psi_*(\mathcal{O}_M) = \mathfrak{c}^{*(0,q)} \otimes \mathcal{O}_N$. Thus, by the Projective Formula,

$$R^q \Psi_*(\wedge^l \Psi^* \Theta_N) = R^q \Psi_*(\Psi^* \wedge^l \Theta_N) = R^q \Psi_*(\mathcal{O}_M) \otimes \wedge^l \Theta_N = \mathfrak{c}^{*(0,q)} \otimes \wedge^l \Theta_N.$$

□

Lemma 4. $H^k(N, \wedge^l \Theta_N) = \mathfrak{t}^{*(0,k)} \otimes \mathfrak{t}^{l,0}$, and $H^k(M, \wedge^l \Psi^* \Theta_N) = \mathfrak{g}^{*(0,k)} \otimes \mathfrak{t}^{l,0}$.

Proof. The proof is essentially the same as that of Lemma 5 in [20]. Here we give a sketch. First note that N is an abelian variety. It follows that Θ_N is holomorphically trivial, and is isomorphic to $\mathfrak{t}^{1,0}$. Therefore, $H^k(N, \wedge^l \Theta_N) = \mathfrak{t}^{*(0,k)} \otimes \mathfrak{t}^{l,0}$.

To compute $H^k(M, \wedge^l \Psi^* \Theta_N)$, we use the standard Leray spectral sequence for the fibration Ψ . By the previous lemma, the second sheet is given by

$$\begin{aligned} E_2^{p,q} &= H^p(N, R^q \Psi_* (\wedge^l \Psi^* \Theta_N)) = H^p(N, \mathfrak{c}^{*(0,q)} \otimes \wedge^l \Theta_N) \\ &= \mathfrak{c}^{*(0,q)} \otimes H^p(N, \wedge^l \Theta_N) = \mathfrak{c}^{*(0,q)} \otimes \mathfrak{t}^{*(0,p)} \otimes \mathfrak{t}^{l,0}. \end{aligned}$$

Thus $d_2 : E_2^{p,q} \rightarrow E_2^{p+2,q-1}$ is a map

$$\mathfrak{c}^{*(0,q)} \otimes \mathfrak{t}^{*(0,p)} \otimes \mathfrak{t}^{l,0} \rightarrow \mathfrak{c}^{*(0,q-1)} \otimes \mathfrak{t}^{*(0,p+2)} \otimes \mathfrak{t}^{l,0}.$$

But d_2 is essentially given by $\bar{\partial}$. Equalities (25) and (26) imply that $d_2 = 0$. It follows that

$$\begin{aligned} H^k(M, \wedge^l \Psi^* \Theta_N) &= \bigoplus_{p+q=k} E_2^{p,q} = \bigoplus_{p+q=k} \mathfrak{c}^{*(0,q)} \otimes \mathfrak{t}^{*(0,p)} \otimes \mathfrak{t}^{l,0} \\ &= \mathfrak{g}^{*(0,k)} \otimes \mathfrak{t}^{l,0}. \end{aligned}$$

□

Lemma 5. *Suppose that*

$$0 \rightarrow Z \rightarrow E \xrightarrow{\rho} Q \rightarrow 0$$

is an exact sequence of holomorphic vector bundles. Then there induces a series of exact sequences

$$\begin{aligned} 0 \rightarrow E^{(1)} &\rightarrow \wedge^l E \rightarrow \wedge^l Q; \\ 0 \rightarrow E^{(2)} &\rightarrow E^{(1)} \rightarrow \wedge^{l-1} Q \otimes Z; \\ &\dots\dots\dots \\ 0 \rightarrow E^{(r+1)} &\rightarrow E^{(r)} \rightarrow \wedge^{l-r} Q \otimes \wedge^r Z; \\ &\dots\dots\dots \\ 0 \rightarrow E^{(l)} &\rightarrow E^{(l-1)} \rightarrow Q \otimes \wedge^{l-1} Z, \end{aligned}$$

where $E^{(l)} = \wedge^l Z$.

Proof. For any $m \geq 0$ and $n \geq 0$, we define a derivative operator $D_\rho : \wedge^m Q \otimes \wedge^n E \rightarrow \wedge^{m+1} Q \otimes \wedge^{n-1} E$ by

$$\begin{aligned} &D_\rho(q_1 \wedge \dots \wedge q_m \otimes e_1 \wedge \dots \wedge e_n) \\ &= \sum_{i=1}^n (-1)^{i-1} q_1 \wedge \dots \wedge q_m \wedge \rho(e_i) \otimes e_1 \wedge \dots \wedge \widehat{e_i} \wedge \dots \wedge e_n. \end{aligned}$$

Now consider

$$D_\rho^l : \wedge^l E \rightarrow \wedge^l Q.$$

Let $E^{(1)} = \text{Ker} D_\rho^l$. Similarly, for each $r = 1, \dots, l$, we consider

$$D_\rho^r : \wedge^l E \rightarrow \wedge^{l+r} Q \otimes \wedge^{l-r} E,$$

and let $E^{(l-r+1)} = \text{Ker} D_\rho^r$.

It is obvious that $\wedge^l Z = E^{(l)} \subset E^{(l-1)} \subset \dots \subset E^{(1)}$. The fact that $E^{(r)}/E^{(r+1)} \cong \wedge^{l-r} Q \otimes \wedge^r Z$ is also easily seen. □

Proof of Theorem 1. Let $E = \Theta_M$, $Z = \mathcal{O}_M \otimes \mathfrak{c}^{1,0}$, $Q = \Psi^* \Theta_N$. Then we have the following exact sequence of holomorphic vector bundles:

$$0 \rightarrow Z \rightarrow E \xrightarrow{\rho} Q \rightarrow 0.$$

By Lemma 5, we have a filtration of $\wedge^l E = \wedge^l \Theta_M$:

$$\mathcal{O}_M \otimes \mathfrak{c}^{l,0} = \wedge^l Z = E^{(l)} \subset E^{(l-1)} \subset \dots \subset E^{(1)} \subset E^{(0)} = \wedge^l E = \wedge^l \Theta_M.$$

Moreover, the associated graded spaces are:

$$\begin{aligned} G^0 &= E^{(0)}/E^{(1)} \cong \wedge^l Q = \Psi^*(\wedge^l \Theta_N); \\ G^1 &= E^{(1)}/E^{(2)} \cong \wedge^{l-1} Q \otimes Z = \Psi^*(\wedge^{l-1} \Theta_N) \otimes \mathfrak{c}^{1,0}; \\ &\dots\dots\dots \\ G^r &= E^{(r)}/E^{(r+1)} = \wedge^{l-r} Q \otimes \wedge^r Z = \Psi^*(\wedge^{l-r} \Theta_N) \otimes \mathfrak{c}^{r,0}; \\ &\dots\dots\dots \\ G^{l-1} &= E^{(l-1)}/E^{(l)} \cong Q \otimes \wedge^{l-1} Z = \Psi^*(\Theta_N) \otimes \mathfrak{c}^{l-1,0}. \end{aligned}$$

Accordingly, we have a filtration of the cochain complex $C^\bullet = T^{*(0,\bullet)} \otimes \wedge^l \Theta_M$:

$$T^{*(0,\bullet)} \otimes \mathfrak{c}^{l,0} = \mathcal{F}^{(l)} C^\bullet \subset \mathcal{F}^{(l-1)} C^\bullet \subset \dots \subset \mathcal{F}^{(1)} C^\bullet \subset \mathcal{F}^{(0)} C^\bullet = C^\bullet,$$

where $\mathcal{F}^{(r)} C^\bullet = T^{*(0,\bullet)} \otimes E^{(r)}$. Thus there associates a spectral sequence: $E_r^{p,q}$, which starts with

$$E_0^{p,q} = \mathcal{F}^{(p)} C^{p+q} / \mathcal{F}^{(p+1)} C^{p+q} = T^{*(0,\bullet)} \otimes G^p \cong T^{*(0,\bullet)} \otimes \Psi^*(\wedge^{l-p} \Theta_N) \otimes \mathfrak{c}^{p,0}$$

and $d_0 = \bar{\partial}$. It follows from Lemma 4 that we have

$$E_1^{p,q} = H^{p+q}(G^p) = H^{p+q}(\Psi^*(\wedge^{l-p} \Theta_N) \otimes \mathfrak{c}^{p,0}) = \mathfrak{g}^{*(0,p+q)} \otimes \mathfrak{t}^{l-p,0} \otimes \mathfrak{c}^{p,0}.$$

The associated $d_1 : E_1^{p,q} \rightarrow E_1^{p+1,q}$ should be exactly

$$\bar{\partial} : \mathfrak{g}^{*(0,p+q)} \otimes \mathfrak{t}^{(l-p,0)} \otimes \mathfrak{c}^{(p,0)} \rightarrow \mathfrak{g}^{*(0,p+1+q)} \otimes \mathfrak{t}^{(l-p-1,0)} \otimes \mathfrak{c}^{(p+1,0)}.$$

Here we use the fact that \mathfrak{g} is 2-step nilpotent, namely the relations in (25) and (26). Hence

$$E_2^{p,q} = H_{d_1}^{p+q}(E_1^{p,q} \rightarrow E_1^{p+1,q}) = \frac{\text{Ker}(\bar{\partial}) \cap \mathfrak{g}^{*(0,p+q)} \otimes \mathfrak{t}^{(l-p,0)} \otimes \mathfrak{c}^{(p,0)}}{\bar{\partial}(\mathfrak{g}^{*(0,p-1+q)} \otimes \mathfrak{t}^{(l-p+1,0)} \otimes \mathfrak{c}^{(p-1,0)})}.$$

The operator $d_2 : E_2^{p,q} \rightarrow E_2^{p+2,q-1}$ is essentially that of $\bar{\partial}$. So the previous expression implies that $d_2 = 0$. So $E_2^{p,q} = E_\infty^{p,q}$. Therefore, one has

$$\begin{aligned} H^k(\wedge^l \Theta_M) &= H_\bar{\partial}^k(C^\bullet) \cong \bigoplus_{p+q=k} E_2^{p,q} \\ &= \bigoplus_{p+q=k} \frac{\text{Ker}(\bar{\partial}) \cap \mathfrak{g}^{*(0,p+q)} \otimes \mathfrak{t}^{l-p,0} \otimes \mathfrak{c}^{p,0}}{\bar{\partial}(\mathfrak{g}^{*(0,p-1+q)} \otimes \mathfrak{t}^{l-p+1,0} \otimes \mathfrak{c}^{p-1,0})} = \bigoplus_p \frac{\text{Ker}(\bar{\partial}) \cap \mathfrak{g}^{*(0,k)} \otimes \mathfrak{t}^{l-p,0} \otimes \mathfrak{c}^{p,0}}{\bar{\partial}(\mathfrak{g}^{*(0,k-1)} \otimes \mathfrak{t}^{l-p+1,0} \otimes \mathfrak{c}^{p-1,0})} \\ &= \frac{\text{Ker}(\bar{\partial}) \cap (\bigoplus_p \mathfrak{g}^{*(0,k)} \otimes \mathfrak{t}^{l-p,0} \otimes \mathfrak{c}^{p,0})}{\bigoplus_p \bar{\partial}(\mathfrak{g}^{*(0,k-1)} \otimes \mathfrak{t}^{l-p+1,0} \otimes \mathfrak{c}^{p-1,0})} = \frac{\text{Ker}(\bar{\partial}) \cap \mathfrak{g}^{*(0,k)} \otimes \mathfrak{g}^{l,0}}{\bar{\partial}(\mathfrak{g}^{*(0,k-1)} \otimes \mathfrak{g}^{l,0})} \\ &= H_\bar{\partial}^k(\mathfrak{g}^{l,0}), \end{aligned}$$

as required. □

3.2 Degeneracy of Holomorphic Poisson Double Complex

As a consequence of Theorem 1,

$$E_2^{p,q} = \frac{\text{kernel of } \text{ad}_\Lambda : H^q(\mathfrak{g}^{p,0}) \rightarrow H^q(\mathfrak{g}^{p+1,0})}{\text{image of } \text{ad}_\Lambda : H^q(\mathfrak{g}^{p-1,0}) \rightarrow H^q(\mathfrak{g}^{p,0})} \quad (27)$$

where $H^q(\mathfrak{g}^{p,0})$ is given in (18).

As an application, we consider the case when the nilmanifold is 2-step with real two-dimensional center. Since

$$\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \mathfrak{c}^{1,0} \quad \text{and} \quad \mathfrak{g}^{*(0,1)} = \mathfrak{t}^{*(0,1)} \oplus \mathfrak{c}^{*(0,1)},$$

and the complex dimension of $\mathfrak{c}^{1,0}$ is equal to one,

$$\mathfrak{g}^{p,0} = \mathfrak{t}^{p,0} \oplus \mathfrak{t}^{p-1,0} \otimes \mathfrak{c}^{1,0}, \quad \mathfrak{g}^{*(0,q)} = \mathfrak{t}^{*(0,q)} \oplus \mathfrak{t}^{*(0,q-1)} \otimes \mathfrak{c}^{*(0,1)}.$$

Lemma 6. $\mathfrak{g}^{*(0,1)} \oplus \mathfrak{c}^{1,0} \subseteq \text{Ker} \bar{\partial}$ and $\bar{\partial} \mathfrak{t}^{1,0} \subseteq \mathfrak{t}^{*(0,1)} \otimes \mathfrak{c}^{1,0}$. Moreover,

$$\bar{\partial}(\mathfrak{t}^{*(0,a)} \otimes \mathfrak{c}^{*(0,b)} \otimes \mathfrak{t}^{k,0} \otimes \mathfrak{c}^{\ell,0}) \subseteq \mathfrak{t}^{*(0,a+1)} \otimes \mathfrak{c}^{*(0,b)} \otimes \mathfrak{t}^{k-1,0} \otimes \mathfrak{c}^{\ell+1,0}. \quad (28)$$

In the subsequent presentation, we suppress the notations for the vector spaces, and simply keep track of the quadruple of indices to indicate the components. With this notation, the statement above is summarized as

$$\bar{\partial}(a, b; k, \ell) \subseteq (a+1, b; k-1, \ell+1). \quad (29)$$

Due to dimensional assumption on $\mathfrak{c}^{1,0}$, when $\ell \geq 1$, then the space $(a+1, b; k-1, \ell+1)$ is trivial.

Next, we turn our attention to the effect of an invariant holomorphic Poisson structure. Given the dimension restriction on $\mathfrak{c}^{1,0}$, $\mathfrak{g}^{2,0} = \mathfrak{c}^{1,0} \otimes \mathfrak{t}^{1,0} \oplus \mathfrak{t}^{2,0}$. With regard to this decomposition, any Λ in $\mathfrak{g}^{2,0}$ is the sum of Λ_1 and Λ_2 where, $\Lambda_1 = W \wedge T$ with $W \in \mathfrak{c}^{1,0}$, $T \in \mathfrak{t}^{1,0}$, and Λ_2 is in $\mathfrak{t}^{2,0}$.

Since the complex structure is abelian, $[\Lambda, \mathfrak{g}^{1,0}] = 0$. One could also see from the structure equations (22) that $[\Lambda, \bar{\omega}^k] = 0$. By (24),

$$[\Lambda_1, \bar{\rho}] = W \wedge [T, \bar{\rho}] \subseteq \mathfrak{t}^{*(0,1)} \otimes \mathfrak{c}^{1,0}. \quad (30)$$

Lemma 7. When $\Lambda = \Lambda_1 + \Lambda_2$, then $\text{ad}_\Lambda(\mathfrak{g}^{1,0}) = 0$, $\text{ad}_\Lambda(\mathfrak{t}^{*(0,1)}) = 0$, and

$$\text{ad}_{\Lambda_1}(\mathfrak{c}^{*(0,1)}) \subseteq \mathfrak{t}^{*(0,1)} \otimes \mathfrak{c}^{1,0}, \quad \text{ad}_{\Lambda_2}(\mathfrak{c}^{*(0,1)}) \subseteq \mathfrak{t}^{*(0,1)} \otimes \mathfrak{t}^{1,0}. \quad (31)$$

As a consequence of the above lemma, $\text{ad}_{\Lambda_1}(\mathfrak{t}^{*(0,a)} \otimes \mathfrak{c}^{*(0,b)} \otimes \mathfrak{t}^{k,0} \otimes \mathfrak{c}^{\ell,0})$ is contained in $\mathfrak{t}^{*(0,a+1)} \otimes \mathfrak{c}^{*(0,b-1)} \otimes \mathfrak{t}^{k,0} \otimes \mathfrak{c}^{\ell+1,0}$, and $\text{ad}_{\Lambda_2}(\mathfrak{t}^{*(0,a)} \otimes \mathfrak{c}^{*(0,b)} \otimes \mathfrak{t}^{k,0} \otimes \mathfrak{c}^{\ell,0})$ is contained in $\mathfrak{t}^{*(0,a+1)} \otimes \mathfrak{c}^{*(0,b-1)} \otimes \mathfrak{t}^{k+1,0} \otimes \mathfrak{c}^{\ell,0}$. Using our shorthand notations, we have

$$\text{ad}_{\Lambda_1}(a, b; k, \ell) \subseteq (a+1, b-1; k, \ell+1), \quad \text{ad}_{\Lambda_2}(a, b; k, \ell) \subseteq (a+1, b-1; k+1, \ell). \quad (32)$$

As the range of ad_{Λ_1} and ad_{Λ_2} are in different spaces,

$$\text{Ker} \text{ad}_\Lambda = \text{Ker} \text{ad}_{\Lambda_1} \cap \text{Ker} \text{ad}_{\Lambda_2}.$$

Theorem 2. Suppose that $M = G/\Gamma$ is a 2-step nilmanifold with an abelian complex structure. Suppose that the center of \mathfrak{g} is real two-dimensional, then for any invariant holomorphic Poisson structure Λ , the Poisson spectral sequence degenerates on the second sheet.

We need to prove that $d_2^{p,q} \equiv 0$ for all p, q . Given the dimension assumption on $\mathfrak{c}^{1,0}$, using our shorthand notation, the space $\mathfrak{g}^{*(0,q)} \otimes \mathfrak{g}^{p,0}$ decomposes into four components:

$$(q, 0; p, 0); \quad (q-1, 1; p, 0); \quad (q, 0; p-1, 1); \quad (q-1, 1; p-1, 1).$$

Suppose that A is in $\mathfrak{g}^{*(0,q)} \otimes \mathfrak{g}^{p,0}$ and $\bar{\partial}A = 0$. Consider $\text{ad}_\Lambda(A) = \text{ad}_{\Lambda_1}A + \text{ad}_{\Lambda_2}A$. By (32), $(q, 0; p, 0)$, $(q, 0; p - 1, 1)$ and $(q - 1, 1; p - 1, 1)$ are in $\text{Ker} \text{ad}_{\Lambda_1}$, and

$$\text{ad}_{\Lambda_1}(A) \in \text{ad}_{\Lambda_1} \mathfrak{g}^{*(0,q)} \otimes \mathfrak{g}^{p,0} = \text{ad}_{\Lambda_1}(q - 1, 1; p, 0) \subseteq (q, 0; p, 1).$$

Similarly, $(q, 0; p, 0)$ and $(q, 0; p - 1, 1)$ are in $\text{Ker} \text{ad}_{\Lambda_2}$, and

$$\begin{aligned} \text{ad}_{\Lambda_2}(A) &\in \text{ad}_{\Lambda_2} \mathfrak{g}^{*(0,q)} \otimes \mathfrak{g}^{p,0} \\ &= \text{ad}_{\Lambda_2}(q - 1, 1; p, 0) \oplus \text{ad}_{\Lambda_2}(q - 1, 1; p - 1, 1) \subseteq (q, 0; p + 1, 0) \oplus (q, 0; p, 1). \end{aligned}$$

$\text{ad}_\Lambda(A)$ represents a zero class in $H^q(\mathfrak{g}^{p+1})$ if and only if there exists B such that $\text{ad}_\Lambda(A) = \bar{\partial}B$. By (29), the image of $\bar{\partial}$ is contained in the component $(a, b; k, 1)$ for some a, b, k . From the above observation on $\text{ad}_{\Lambda_1}(A) + \text{ad}_{\Lambda_2}(A)$, such B exists only if $\text{ad}_\Lambda(A) \in (q, 0; p, 1)$. Furthermore by (29) again, B is contained in the component $(q - 1, 0; p + 1, 0)$; which is contained in $\mathfrak{t}^{*(0,q-1)} \otimes \mathfrak{t}^{p+1,0}$. Then d_2A is represented by $\text{ad}_\Lambda(B)$. However, by Lemma 7, $(q - 1, 0; p + 1, 0)$ is in the kernel of both ad_{Λ_1} and ad_{Λ_2} . Therefore, $d_2 \equiv 0$ as claimed.

3.3 Examples

We consider examples of holomorphic Poisson structures that satisfy the conditions in Theorem 2. In particular, let W be an element in $\mathfrak{t}^{1,0}$ and T an element in $\mathfrak{t}^{1,0}$. Define $\Lambda = W \wedge T$. Since $\bar{\partial}(0, 0; 1, 1) \subseteq (1, 0; 0, 2)$ by (29), the dimension restriction implies that Λ is holomorphic. As the complex structure is abelian, Λ is Poisson.

The first example of non-abelian nilmanifold is the Kodaira surface. As a nilmanifold, it is covered by the product of the real three-dimensional Heisenberg group H_3 and the one-dimensional trivial additive group R^1 . Computation on this manifold was done extensively in [23]. It is known to have an invariant holomorphic symplectic structure. Using the usual transformation from symplectic structure to Poisson structure, one obtains a holomorphic Poisson structure on such a Kodaira surface such that the Poisson bi-vector field is of the type $W \wedge T$, and hence satisfies the conditions in Theorem 2. We skip the details for this case, and refer readers to [23].

Six-dimensional 2-step nilmanifolds with abelian complex structures were studied extensively in [6] [20] [27]. The underlying algebraic structure is classified. In particular, let R^n denotes the n -dimensional abelian group and H_n the n -dimensional real Heisenberg group, then the 2-step nilpotent groups with abelian complex structures are R^6 , $H_3 \times R^3$, $H_5 \times R^1$, $H_3 \times H_3$, the real six-dimensional Iwasawa group W_6 and one additional group P_6 which we will describe with a little more details below. Since R^6 is abelian and $H_3 \times R^3$ covers the product of a Kodaira surface and an elliptic curve, they are not new examples in a strict sense. We consider the four remaining cases in their respective series in all admissible dimensions, namely $H_{2n+1} \times R^1$, $H_{2n+1} \times H_{2m+1}$, W_{4n+2} and P_{4n+2} . We demonstrate that they will each provide an example for Theorem 2.

3.3.1 $H_{2n+1} \times R^1$

Let $\{X_k, Y_k, Z_1, Z_2 : 1 \leq k \leq n\}$ be a basis of a $(2n + 2)$ -dimensional vector space \mathfrak{g} . Define a Lie bracket by

$$\begin{aligned} \llbracket X_k, Y_k \rrbracket &= -\llbracket Y_k, X_k \rrbracket = Z_1, \quad \llbracket X_k, Z_1 \rrbracket = \llbracket Y_k, Z_1 \rrbracket = 0, \\ \llbracket X_k, Z_2 \rrbracket &= \llbracket Y_k, Z_2 \rrbracket = \llbracket Z_1, Z_2 \rrbracket = 0 \end{aligned} \tag{33}$$

for all k . Then $\{X_k, Y_k, Z_1 : 1 \leq k \leq n\}$ span the $(2n + 1)$ -dimensional Heisenberg algebra \mathfrak{h}_{2n+1} . With Z_2 , \mathfrak{g} is the one-dimensional trivial extension of the Heisenberg algebra. Its center is spanned by Z_1 and Z_2 . Now, define a linear map by

$$JX_k = Y_k, \quad JY_k = -X_k, \quad JZ_1 = Z_2, \quad JZ_2 = -Z_1.$$

It defines an abelian complex structure.

3.3.2 $H_{2n+1} \times H_{2m+1}$

Let $\{X_k, Y_k, Z_1 : 1 \leq k \leq n\}$ be a basis for the $(2n + 1)$ -dimensional Heisenberg algebra \mathfrak{h}_{2n+1} as in (33). Let $\{U_\ell, V_\ell, Z_2 : 1 \leq \ell \leq m\}$ be a basis for a $(2m + 1)$ -dimensional Heisenberg algebra \mathfrak{h}_{2m+1} so that $\llbracket U_\ell, V_\ell \rrbracket = Z_2$ for all ℓ . On $\mathfrak{h}_{2n+1} \oplus \mathfrak{h}_{2m+1}$, its center is spanned by Z_1 and Z_2 . Consider the linear map

$$JX_k = Y_k, \quad JY_k = -X_k, \quad JU_\ell = V_\ell, \quad JV_\ell = -U_\ell, \quad JZ_1 = Z_2, \quad JZ_2 = -Z_1.$$

It defines an abelian complex structure.

3.3.3 W_{4n+2}

It is a $(4n + 2)$ -dimensional nilpotent group. On Lie algebra level, let

$$\{Z_1, Z_2, X_{4k+1}, X_{4k+2}, X_{4k+3}, X_{4k+4} : 0 \leq k \leq n - 1\} \quad (34)$$

be a basis such that the non-zero structure equations are given by

$$\begin{aligned} \llbracket X_{4k+1}, X_{4k+3} \rrbracket &= -\frac{1}{2}Z_1, & \llbracket X_{4k+1}, X_{4k+4} \rrbracket &= -\frac{1}{2}Z_2, \\ \llbracket X_{4k+2}, X_{4k+3} \rrbracket &= -\frac{1}{2}Z_2, & \llbracket X_{4k+2}, X_{4k+4} \rrbracket &= \frac{1}{2}Z_1. \end{aligned} \quad (35)$$

We obtain an abelian complex structure if we insist $J^2 = -\text{identity}$, and

$$JX_{4k+1} = X_{4k+2}, \quad JX_{4k+3} = -X_{4k+4}, \quad JZ_1 = -Z_2. \quad (36)$$

3.3.4 P_{4n+2}

It is again a $(4n + 2)$ -dimensional nilpotent group. On Lie algebra level, it is a degeneration of the structure in (35). In particular, with respect to the basis as given in (34), the non-zero structure equations are given by

$$\llbracket X_{4k+1}, X_{4k+2} \rrbracket = -\frac{1}{2}Z_1, \quad \llbracket X_{4k+1}, X_{4k+4} \rrbracket = -\frac{1}{2}Z_2, \quad \llbracket X_{4k+2}, X_{4k+3} \rrbracket = -\frac{1}{2}Z_2.$$

An abelian complex structure is defined by the requirements in (36).

4 Complex Parallelizable Nilmanifolds

A nilmanifold is complex parallelizable if there is an invariant complex structure J such that for all X, Y in \mathfrak{g} ,

$$\llbracket X, JY \rrbracket = J\llbracket X, Y \rrbracket. \quad (37)$$

In this case, \mathfrak{g} is the underlying real Lie algebra of a complex Lie algebra, and

$$\llbracket \mathfrak{g}^{1,0}, \mathfrak{g}^{1,0} \rrbracket \subseteq \mathfrak{g}^{1,0}, \quad \llbracket \mathfrak{g}^{1,0}, \mathfrak{g}^{0,1} \rrbracket = \{0\}.$$

Dually, when ∂ is the Chevalley-Eilenberg differential on $\mathfrak{g}^{*(1,0)}$, then $(\oplus \mathfrak{g}^{*(p,0)}, \partial)$ forms an exterior differential algebra. So is the conjugate. Let $H_\partial^q(\mathfrak{g}^{*(0,1)})$ denote the q -th cohomology of the differential algebra $(\oplus \mathfrak{g}^{*(0,q)}, \bar{\partial})$.

It is proved in [26] that the Dolbeault cohomology $H^{p,q}(M)$ is isomorphic to the $H_\partial^q(\mathfrak{g}^{*(0,1)}) \otimes \mathfrak{g}^{*(p,0)}$.

Since the holomorphic tangent bundle for a complex parallelizable manifold is trivialized by the complex vector fields of the algebra $\mathfrak{g}^{1,0}$,

$$H^q(M, \Theta^p) \cong H^{0,q}(M) \otimes \mathfrak{g}^{p,0} \cong H_\partial^q(\mathfrak{g}^{*(0,1)}) \otimes \mathfrak{g}^{p,0}. \quad (38)$$

Suppose that Λ is an invariant holomorphic Poisson structure on M . The above observation implies that the first sheet of the Poisson double complex is given by

$$E_1^{p,q} = H_\partial^q(\mathfrak{g}^{*(0,1)}) \otimes \mathfrak{g}^{p,0}. \tag{39}$$

Since the exterior differential of $\mathfrak{g}^{*(0,1)}$ is contained in $\mathfrak{g}^{*(0,2)}$, $\text{ad}_\Lambda(\bar{\omega}) = 0$ for all $\bar{\omega} \in \mathfrak{g}^{*(0,1)}$. If $\sum_j \bar{\mathcal{D}}^j \otimes \theta_j$ represents an element in $H_\partial^q(\mathfrak{g}^{*(0,1)}) \otimes \mathfrak{g}^{p,0}$ where $\bar{\mathcal{D}}^j$ for appropriate number of j forms a basis for $H_\partial^q(\mathfrak{g}^{*(0,1)})$, and $\theta_j \in \mathfrak{g}^{p,0}$, then

$$\text{ad}_\Lambda(\sum_j \bar{\mathcal{D}}^j \otimes \theta_j) = \sum_j (-1)^q \bar{\mathcal{D}}^j \otimes \text{ad}_\Lambda(\theta_j). \tag{40}$$

Therefore, it represents a zero class in $H_\partial^{q+1}(\mathfrak{g}^{*(0,1)}) \otimes \mathfrak{g}^{p,0}$ if and only if $\text{ad}_\Lambda(\theta_j) = 0$ for each j . It follows that $d_2^{p,q} \equiv 0$. Moreover, $(\mathfrak{g}^{p,0}, \text{ad}_\Lambda)$ form a complex. Let $H_{\text{ad}_\Lambda}^p(\mathfrak{g}^{1,0})$ be the p -th cohomology, then

$$E_2^{p,q} = H_\partial^q(\mathfrak{g}^{*(0,1)}) \otimes H_{\text{ad}_\Lambda}^p(\mathfrak{g}^{1,0}).$$

Due to the degeneracy, $H_\Lambda^k(M) = \oplus_{p+q=k} H_\partial^q(\mathfrak{g}^{*(0,1)}) \otimes H_{\text{ad}_\Lambda}^p(\mathfrak{g}^{1,0})$. In summary, we have

Theorem 3. *Let M be a complex parallelizable manifold with an invariant holomorphic Poisson structure Λ . Let $\mathfrak{g}^{(1,0)}$ be the vector space of invariant holomorphic vector fields on M . Let $H_\partial^q(\mathfrak{g}^{*(0,1)})$ be the Chevalley-Eilenberg cohomology for the conjugation algebra $\mathfrak{g}^{0,1}$, and $H_{\text{ad}_\Lambda}^p(\mathfrak{g}^{1,0})$ the cohomology of the operator $\text{ad}_\Lambda = [\Lambda, -]$ on the exterior algebra of $\mathfrak{g}^{1,0}$, then*

$$H_\Lambda^k(M) = \oplus_{p+q=k} H_\partial^q(\mathfrak{g}^{*(0,1)}) \otimes H_{\text{ad}_\Lambda}^p(\mathfrak{g}^{1,0}).$$

4.1 Complex Three-dimensional Iwasawa Manifolds

Let G be the complex Heisenberg group of 3×3 matrices of the form

$$\mathbf{z} := \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let H be the co-compact lattice of matrices whose entries are Gaussian integers. As a complex manifold, $G \simeq \mathbb{C}^3$. Let $\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}$ be the corresponding coordinate global vector fields. Viewed as matrices,

$$\frac{\partial}{\partial z_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{\partial}{\partial z_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{\partial}{\partial z_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, $\mathfrak{g}^{1,0}$ is spanned by the vector fields

$$W_1 = \frac{\partial}{\partial z_1}, \quad W_2 = \frac{\partial}{\partial z_2}, \quad W_3 = \frac{\partial}{\partial z_3} + z_1 \frac{\partial}{\partial z_2}$$

which are all holomorphic. Their dual basis is given by

$$\omega_1 = dz_1, \quad \omega_2 = dz_2 - z_1 dz_3, \quad \omega_3 = dz_3$$

and $\bar{\partial}\bar{\omega}_1 = \bar{\partial}\bar{\omega}_3 = 0, \bar{\partial}\bar{\omega}_2 = -\bar{\omega}_1 \wedge \bar{\omega}_3$. Therefore,

$$E_1^{0,0} = \langle 1 \rangle, \quad E_1^{0,1} = \langle \bar{\omega}_1, \bar{\omega}_3 \rangle \quad E_1^{0,2} = \langle \bar{\omega}_1 \wedge \bar{\omega}_2, \bar{\omega}_2 \wedge \bar{\omega}_3 \rangle, \quad E_1^{0,3} = \langle \bar{\omega}_1 \wedge \bar{\omega}_2 \wedge \bar{\omega}_3 \rangle.$$

Also note that

$$[W_1, W_2] = [W_2, W_3] = 0, \quad [W_1, W_3] = W_2$$

and $[W_i, \bar{w}_j] = 0$. By direct computation, the invariant holomorphic Poisson structures are given by

$$\Lambda = aW_1 \wedge W_2 + bW_2 \wedge W_3,$$

for some constants $a, b \in \mathbb{C}$. Moreover, as we get $\text{ad}_\Lambda = 0, d_1 = 0$:

Proposition 3. *Let M be the Iwasawa manifold, equipped with the usual parallelizable complex structure. If Λ is any invariant holomorphic Poisson structure on M , then the corresponding spectral sequence degenerates at the first step, and $H_\Lambda^k(M) = \bigoplus_{p+q=k} H_\mathbb{C}^q(\mathfrak{g}^{*(0,1)}) \otimes \mathfrak{g}^{1,0}$.*

5 Lichnerowicz Homomorphism

On any complex manifold, there is a well known double complex, the Dolbeault bi-complex. Recall that $d = \partial + \bar{\partial}$. Define

$$B^{p,q} = C^\infty(M, \wedge^p T^{*(1,0)}M \otimes \wedge^q T^{*(0,1)}M).$$

Due to the integrability of the complex structure, we have the double complex

$$\begin{array}{ccc} B^{p,q+1} & \longrightarrow & B^{p+1,q+1} \\ \bar{\partial} \uparrow & & \uparrow \\ B^{p,q} & \longrightarrow & B^{p+1,q} \\ & & \partial \end{array} .$$

Recall that $A^{p,q} = C^\infty(M, \wedge^p T^{(1,0)}M \otimes \wedge^q T^{*(0,1)}M)$. Given a holomorphic Poisson structure Λ , contraction with $-\Lambda$ defines a map from $B^{1,0}$ to $A^{2,0}$. We extend to the map $\phi : B^{p,q} \rightarrow A^{p,q}$ as follows.

1. $\phi(\omega) = -(\Lambda\omega)$ for all ω in $B^{1,0}$.
2. $\phi(\bar{w}) = \bar{w}$ for all \bar{w} in $B^{0,1}$.
3. $\phi = \text{identity}$ on $B^{0,0}$.
4. Extend ϕ to $B^{p,q}$ for $p, q \geq 1$ by $\phi(A \wedge B) = \phi(A) \wedge \phi(B)$.

By construction, it is apparent that ϕ maps $B^{p,q}$ to $A^{p,q}$. The rest of this section is to prove the following observation.

Theorem 4. *The map ϕ is a homomorphism from the Dolbeault double complex to the Poisson double complex.*

$$\bar{\partial} \circ \phi = \phi \circ \bar{\partial}, \quad \text{ad}_\Lambda \circ \phi = \phi \circ \partial. \tag{41}$$

It suffices to verify the above identities on $B^{0,0}, B^{1,0}$ and $B^{0,1}$. For any function $f, \bar{\partial} \circ \phi f = \bar{\partial} f$. As $\bar{\partial} f$ is a $(0, 1)$ -form, it is clear that $\phi \circ \bar{\partial} f = \bar{\partial} f$. To work on the other identity, we do it locally, and there exists $(1, 0)$ holomorphic vector fields X_j and Y_j such that at least locally, $\Lambda = \sum_j X_j \wedge Y_j$. Then

$$\phi \partial f = - \sum_j (X_j \wedge Y_j) df = - \sum_j ((X_j f) Y_j - (Y_j f) X_j).$$

On the other hand,

$$\text{ad}_\Lambda(\phi f) = \sum_j \llbracket X_j \wedge Y_j, f \rrbracket = \sum_j (X_j \wedge \llbracket Y_j, f \rrbracket - Y_j \wedge \llbracket X_j, f \rrbracket) = \sum_j ((Y_j f) X_j - (X_j f) Y_j).$$

Therefore, the identities in (41) are satisfied when they are applied to smooth functions on the manifold.

Next, we test the identities (41) against $(0, 1)$ -forms, say \bar{w} . Since the map ϕ is the identity map when it is restricted to the bundle of $(0, q)$ -forms. Therefore, the identity $\bar{\partial} \circ \phi(\bar{w}) = \phi(\bar{\partial} \bar{w})$ is obviously satisfied.

Lemma 8. For any $(2, 0)$ -form Ω , $(1, 0)$ -forms γ and δ , $\phi(\Omega)(\gamma, \delta) = \Omega(\Lambda\gamma, \Lambda\delta)$. For any $(1, 1)$ -form Γ , $(1, 0)$ -form γ and $(0, 1)$ -vector \bar{Z} , $\phi(\Gamma)(\gamma, \bar{Z}) = \Gamma(\Lambda\gamma, \bar{Z})$.

Proof. By linearity, it suffices to consider the case when $\Omega = \alpha \wedge \beta$ where α and β are $(1, 0)$ -forms. In this case, it is just a matter of definition of ϕ because,

$$\phi(\alpha \wedge \beta)(\gamma, \delta) = (\Lambda(\alpha) \wedge \Lambda(\beta))(\gamma, \delta) = \Lambda(\alpha, \gamma)\Lambda(\beta, \delta) - \Lambda(\alpha, \delta)\Lambda(\beta, \gamma) = \alpha(\Lambda\gamma)\beta(\Lambda\delta) - \alpha(\Lambda\delta)\beta(\Lambda\gamma).$$

The proof of the second statement is similar. \square

Suppose that γ and \bar{Z} are as given in the last lemma, then

$$\phi(\partial\bar{\omega})(\gamma, \bar{Z}) = \partial\bar{\omega}(\Lambda\gamma, \bar{Z}) = (\Lambda\gamma)(\bar{\omega}(\bar{Z}) - \bar{Z}(\bar{\omega}(\Lambda\gamma)) - \bar{\omega}(\llbracket\Lambda\gamma, \bar{Z}\rrbracket)) = \mathcal{L}_{\Lambda\gamma}\bar{\omega}(\bar{Z}) - \bar{\omega}(\mathcal{L}_{\Lambda\gamma}\bar{Z}) = (\mathcal{L}_{\Lambda\gamma}\bar{\omega})\bar{Z}.$$

It is equal to $\bar{Z}(\llbracket\Lambda\gamma, \bar{\omega}\rrbracket) = \text{ad}_\Lambda(\bar{\omega})(\gamma, \bar{Z})$ as a consequence of the observation of the next lemma.

Lemma 9. [24, Lemma 4, Page 10] For any $(1, 0)$ -vector field X , $(0, 1)$ -vector field \bar{Z} , $(0, 1)$ -form $\bar{\omega}$ and $(1, 0)$ -forms γ , and δ ,

$$\begin{aligned} \llbracket\Lambda, X\rrbracket(\gamma, \delta) &= -\Lambda(\omega)(\gamma(\Lambda\delta)) - \gamma(\llbracket\Lambda\delta, X\rrbracket) + \delta(\llbracket\Lambda\gamma, X\rrbracket), \\ \llbracket\Lambda, \bar{\omega}\rrbracket(\gamma, \bar{Z}) &= \bar{Z}(\llbracket\Lambda\gamma, \bar{\omega}\rrbracket). \end{aligned}$$

Since the identities (41) hold when they are tested on $(0, 1)$ -forms, we next test them on a $(1, 0)$ -form ω . In particular, for any $(1, 0)$ -forms ω , γ , and δ ,

$$\phi(\partial\omega)(\gamma, \delta) = (\partial\omega)(\Lambda\gamma, \Lambda\delta) = (\Lambda\gamma)(\omega(\Lambda\delta)) - (\Lambda\delta)(\omega(\Lambda\gamma)) - \omega(\llbracket\Lambda\gamma, \Lambda\delta\rrbracket). \quad (42)$$

Recall the following observation.

Lemma 10. [24, Lemma 18, Page 31] Let Λ be a bi-vector field. For any $(1, 0)$ -forms ω , γ , and δ ,

$$\frac{1}{2}\llbracket\Lambda, \Lambda\rrbracket(\omega, \gamma, \delta) = \omega(\llbracket\Lambda\gamma, \Lambda\delta\rrbracket) - (\Lambda\omega)(\Lambda(\gamma, \delta)) + \gamma(\llbracket\Lambda\delta, \Lambda\omega\rrbracket) - (\Lambda\gamma)(\Lambda(\delta, \omega)) + \delta(\llbracket\Lambda\omega, \Lambda\gamma\rrbracket) - (\Lambda\delta)(\Lambda(\omega, \gamma)).$$

Since the bivector field Λ is Poisson, $\llbracket\Lambda, \Lambda\rrbracket = 0$. By the last lemma, the identity (42) is equal to

$$-(\Lambda\omega)(\Lambda(\gamma, \delta)) + \gamma(\llbracket\Lambda\delta, \Lambda\omega\rrbracket) + \delta(\llbracket\Lambda\omega, \Lambda\gamma\rrbracket).$$

By Lemma 9, it is equal to $-\llbracket\Lambda, \Lambda\omega\rrbracket(\gamma, \delta)$. By definition of ϕ , it is equal to $\text{ad}_\Lambda(\phi(\omega))(\gamma, \delta)$. As this is true for all γ and δ , $\text{ad}_\Lambda(\phi(\omega)) = \phi(\partial\omega)$ as needed.

By Lemma 8, for any $(1, 0)$ -forms ω , γ and $(0, 1)$ -vector field \bar{Z} ,

$$\begin{aligned} \phi(\bar{\partial}\omega)(\gamma, \bar{Z}) &= (\bar{\partial}\omega)(\Lambda\gamma, \bar{Z}) = (d\omega)(\Lambda\gamma, \bar{Z}) \\ &= (\Lambda\gamma)\omega(\bar{Z}) - \bar{Z}(\omega(\Lambda\gamma)) - \omega(\llbracket\Lambda\gamma, \bar{Z}\rrbracket) = -\bar{Z}(\Lambda(\gamma, \omega)) - \omega(\llbracket\Lambda\gamma, \bar{Z}\rrbracket) \\ &= \bar{Z}(\Lambda(\omega, \gamma)) + \omega(\llbracket\bar{Z}, \Lambda\gamma\rrbracket). \end{aligned} \quad (43)$$

On the other hand, we treat $\bar{\partial}$ as the Lie algebroid differential for the bundle \bar{L} , then

$$\begin{aligned} (\bar{\partial}\phi\omega)(\gamma, \bar{Z}) &= -(\bar{\partial}(\Lambda\omega))(\gamma, \bar{Z}) \\ &= -\rho(\omega)((\Lambda\omega)(\bar{Z})) + \rho(\bar{Z})\omega(\Lambda\omega) + \Lambda\omega(\llbracket\omega, \bar{Z}\rrbracket) \\ &= \bar{Z}(\omega(\Lambda\omega)) - (\mathcal{L}_{\bar{Z}}\omega)\Lambda\omega = \bar{Z}(\omega(\Lambda\omega)) - \bar{Z}(\omega(\Lambda\omega)) + \omega(\llbracket\bar{Z}, \Lambda\omega\rrbracket) \\ &= \omega(\llbracket\bar{Z}, \Lambda\omega\rrbracket). \end{aligned} \quad (44)$$

As a consequence of Lemma 1 and the assumption that Λ is holomorphic, for any $(1, 0)$ -forms ω , γ and $(0, 1)$ -vector field \bar{Z} ,

$$\bar{Z}(\Lambda(\omega, \gamma)) + \omega(\llbracket\bar{Z}, \Lambda\gamma\rrbracket) = \gamma(\llbracket\bar{Z}, \Lambda\omega\rrbracket). \quad (45)$$

By (43) and (44), we conclude that the identities in (41) hold when tested against any $(1, 0)$ -forms.

The proof of Theorem 4 is now complete.

5.1 Applications

The first identity in (41) implies that the bundle map ϕ induces a complex linear homomorphism

$$\phi : H^q(X, \Omega^p) \longrightarrow H^q(X, \Theta^p).$$

The second identity implies that this homomorphism fits into the following commutative diagram:

$$\begin{array}{ccc} & d_1^{p,q} = ad_\Lambda & \\ H^q(X, \Theta^p) & \longrightarrow & H^q(X, \Theta^{p+1}) \\ \phi \uparrow & & \uparrow \phi \\ H^q(X, \Omega^p) & \longrightarrow & H^q(X, \Omega^{p+1}) \\ & d_1^{p,q} = \partial & \end{array}$$

where Ω^p is the holomorphic bundle of $(p, 0)$ -forms. The map is defined so that when $p = 0$, ϕ is an identity map.

Lemma 11. *Suppose that X is holomorphic Poisson manifold such that the Frölicher spectral sequence degenerates at the E_1 -level. Then, the spectral sequence of the holomorphic Poisson double complex satisfies the conditions*

$$E_n^{0,q} = E_1^{0,q} = H^q(X, \mathcal{O}),$$

and $d_n^{0,q} \equiv 0$ for all q and for all $n \geq 2$.

Proof. As noted above, when $p = 0$, the homomorphism ϕ is an isomorphism. The $E_1^{0,q}$ -term for the Frölicher spectral sequence is $H^q(X, \mathcal{O})$. When $d_1^{p,q} = \partial$ is identically equal to zero on it, then $d_1^{0,q} = ad_\Lambda$ is identically equal to zero on the $E_1^{0,q}$ -term for the Poisson spectral sequence, which is $H^q(X, \mathcal{O})$ as well. It follows that $E_2^{0,q} = H^q(X, \mathcal{O})$.

Suppose that $\bar{\omega}$ is a $\bar{\partial}$ -closed $(0, q)$ -form representing an element in $H^q(X, \mathcal{O})$. Since $d_1\bar{\omega}$ represents the zero element in $H^q(X, \Theta)$, there exists a section A of $A^{1,q-1}$ such that

$$ad_\Lambda(\bar{\omega}) = \bar{\partial}A.$$

By definition, $d_2^{0,q}[\bar{\omega}]$ is represented by $ad_\Lambda A$. This is an element in $A^{2,q-1}$, representing a class in $H^{q-1}(X, \Theta^2)$. However, since $\phi(\bar{\omega}) = \bar{\omega}$, we also have

$$\bar{\partial}A = ad_\Lambda(\bar{\omega}) = ad_\Lambda(\phi(\bar{\omega})) = \phi(\bar{\partial}\bar{\omega}) = 0.$$

It follows that $ad_\Lambda(\bar{\omega}) = \bar{\partial}A = 0$. By choosing $A = 0$, we find that $d_2^{0,q}[\bar{\omega}]$ is represented by zero. Therefore, $d_2^{0,q} \equiv 0$ for all q .

For $n \geq 2$ and for all q , since

$$E_{n+1}^{0,q} = \text{Ker } d_n^{0,q} : E_n^{0,q} \rightarrow E_n^{n,q-n+1}.$$

It is now apparent that by induction, $E_{n+1}^{0,q} = E_n^{0,q}$. After all, we have seen that if $\bar{\omega}$ is in $H^q(X, \mathcal{O})$, then under the current assumption, $\bar{\partial}\bar{\omega} + ad_\Lambda(\bar{\omega}) = 0$. □

Theorem 5. *The holomorphic Poisson spectral sequence on any compact holomorphic Poisson surface degenerates at E_2 -level.*

Proof. The first sheet of the holomorphic Poisson spectral sequence of a compact complex surface is:

$$\begin{array}{ccccccc} H^2(\mathcal{O}) & \xrightarrow{d_1^{0,2}} & H^2(\Theta) & \xrightarrow{d_1^{1,2}} & H^2(\Theta^2) & \longrightarrow & 0 \\ H^1(\mathcal{O}) & \xrightarrow{d_1^{0,1}} & H^1(\Theta) & \xrightarrow{d_1^{1,1}} & H^1(\Theta^2) & \longrightarrow & 0 \cdot \\ H^0(\mathcal{O}) & \xrightarrow{d_1^{0,0}} & H^0(\Theta) & \xrightarrow{d_1^{1,0}} & H^0(\Theta^2) & \longrightarrow & 0 \end{array}$$

Since the Frölicher spectral sequence degenerates at E_1 -level on any compact complex surface, the previous lemma is applicable. Therefore, the second sheet of the holomorphic Poisson spectral sequence is given as below.

$$\begin{array}{ccccccc} H^2(\mathcal{O}) = E_2^{0,2} & & E_2^{1,2} & & E_2^{2,2} & & 0 \\ & & \searrow & & & & \\ H^1(\mathcal{O}) = E_2^{0,1} & & E_2^{1,1} & & E_2^{2,1} & & 0 \\ & & \searrow & & & & \\ H^0(\mathcal{O}) = E_2^{0,0} & & E_2^{1,0} & & E_2^{2,0} & & 0 \end{array} .$$

Due to a dimension restriction, the only non-trivial differentials on the second sheet are

$$d_2^{0,1} : H^1(\mathcal{O}) = E_2^{0,1} \longrightarrow E_2^{2,0} \quad \text{and} \quad d_2^{0,2} : H^2(\mathcal{O}) = E_2^{0,2} \longrightarrow E_2^{2,1} .$$

By the second part of the previous lemma, these two maps are identically zero. Therefore, $d_2^{p,q} \equiv 0$ for all (p, q) . \square

Theorem 6. *Suppose that X is holomorphic Poisson manifold with complex dimension n . If the Frölicher spectral sequence of its complex structure degenerates at the E_1 -level, then its holomorphic Poisson spectral sequence degenerates at the E_n -level.*

Proof. Similar to the case when $n = 2$, on a complex manifold with dimension n , the only non-trivial terms at the top sheet are

$$d_n^{0,n-1} : E_n^{0,n-1} \longrightarrow E_n^{n,0} \quad \text{and} \quad d_n^{0,n} := E_n^{0,n} \longrightarrow E_n^{n,1} .$$

\square

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