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Self-Dual Manifolds with Symmetry

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ABSTRACT. An oriented Riemannian four-manifold is said to be *half-conformally-flat* if its conformal curvature W is either self-dual or anti-self-dual as a bundle-valued two-form. We review a construction **[18]** of compact half-conformally-flat manifolds with semifree isometric S^1-action, starting from the Green's function of a collection of points in a hyperbolic three-manifold. If the three-manifold in question is just hyperbolic space, the resulting four-manifolds are one-point conformal compactifications of scalar-flat Kähler surfaces. We then show that any asymptotically Euclidean scalar-flat Kähler surface with a nonzero conformal Killing field arises from this construction.

1. Overview

Imagine that you are given a smooth oriented compact four-manifold M, and, perhaps motivated by pleasant memories of the uniformization theory of surfaces, you try to find a conformally-flat metric g on it. You would thus be looking for a Riemannian metric g which, relative to some atlas of charts $\{x_\alpha : \mathscr{U}_\alpha \longrightarrow \mathscr{V}_\alpha \subset \mathbb{R}^4\}$ for M, is of the form

$$g = f_\alpha \sum_{j=1}^{4} (dx_\alpha^j)^{\otimes 2}$$

on each open set $\mathscr{U}_\alpha$, where f_α is some positive smooth function. Generally speaking, you would be out of luck, for the following reason: the conformal curvature[1] W of such a metric must vanish, whereas, for an arbitrary Riemannian metric g on M, the signature $\sigma = b_+ - b_-$ of M is given by

$$\sigma = \frac{1}{12\pi^2} \int_M (\|W_+\|^2 - \|W_-\|^2)\, d\,\mathrm{vol},$$

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[1]Weyl tensor: the totally-trace-free piece of the Riemann curvature.

so the existence of a conformally-flat metric on M implies that $\sigma = 0$. Here $W_{\pm}$ is the self-dual (respectively, anti-self-dual) piece of the Weyl tensor, as defined by $W_{\pm} := \frac{1}{2}(W \pm *W)$, where the Hodge-star operator $*$ treats W as a bundle-valued two-form.

Since the above problem is, generally speaking, hopeless, one should perhaps try to settle for a metric with as little conformal curvature W as allowed by the topology. A good measure of the the total amount of conformal curvature present is given by the L^2-norm

$$A(g) := \int_M \|W\|^2 \, d\text{ vol} = \int_M (\|W_+\|^2 + \|W_-\|^2) \, d\text{ vol}$$

of the Weyl tensor, because, unlike other norms you might be tempted to apply, this one has the virtue that it is *conformally invariant* — $A(g) = A(f \cdot g)$ for any smooth positive function f. But $A(g) \geq 12\pi^2|\sigma|$, with equality iff either $W_+ \equiv 0$ or $W_- \equiv 0$, in which case the conformal Riemannian manifold $(M, [g])$ is said to be *half-conformally-flat.* The cases $W_+ \equiv 0$ and $W_- \equiv 0$ are more specifically called anti-self-dual and self-dual, respectively; they differ, of course, only by a choice of the orientation.

The two simplest examples of compact *self-dual* manifolds are provided by the Riemannian symmetric spaces S^4 and $\mathbb{CP}_2$, oriented in the usual manner. The half-conformal-flatness of the latter example has, fundamentally, nothing at all to do with its Kähler structure, to such an extent that the following general observation now seem rather surprising: a Kähler manifold of complex dimension 2 is *anti-self-dual* with respect to the standard orientation iff its scalar curvature is identically zero **[8]**.

A particularly compelling reason for the study of half-conformally-flat four-manifolds comes from the Penrose twistor construction **[2, 24]**. Let (M, g) be an orientable Riemannian four-manifold, and let $F \to M$ be the principal $SO(4)$-bundle of orthonormal frames determining the same orientation on M. Let $Z = F/U(2)$, which is a bundle over M with typical fiber $S^2 = SO(4)/U(2)$. Then the smooth six-manifold Z carries a natural almost-complex structure $J : TZ \to TZ$, $J^2 = -1$, which leaves invariant both the tangent spaces of each fiber and the horizontal spaces of the metric connection of g. Indeed, let us notice that, by construction, Z is exactly the space of almost-complex structures $\jmath : TM \to TM$ compatible with the given metric and orientation, and so, thinking of the g-horizontal subspace of TZ as the pull-back of TM to Z, there is thus a tautological way to let J act on the horizontal sub-bundle of TZ. In the vertical directions, on the other hand, J will simply act as the standard complex structure on S^2, namely rotation by $+90^\circ$. Provided that we give the fibers the correct orientation in defining this almost-complex structure J, the entire construction turns out, rather surprisingly, to be *conformally invariant*, meaning that J remains completely unchanged if the given Riemannian metric g is replaced by αg, where $\alpha : M \to \mathbb{R}^+$ is any smooth positive function. This construction of an almost-complex manifold for each conformal Riemannian manifold may thus be thought of as a higher-dimensional analogue of the correspondence between conformal Riemannian two-manifolds and complex one-manifolds. However, the almost-complex manifold (Z, J) will not, in general be a complex manifold— there need not be an atlas of charts for Z relative to which J identically becomes multiplication by i in $\mathbb{C}^3 = \mathbb{R}^6$. Instead, the relevant integrability condition turns out to $W_+ = 0$. When Z is the space of almost-complex structures on M compatible with the given metric and the *conjugate* orientation on M, the tautological almost complex structure on the twistor space is integrable if and only if $W_- = 0$. In short, every half-conformally-flat four-manifold determines a complex three-fold Z, called its *twistor space*, and this complex three-manifold in turn completely encodes the conformal geometry of the original manifold. For example, the twistor space of S^4 is $\mathbb{CP}_3$, whereas the twistor space of $\mathbb{CP}_2$ is the flag-manifold

$$\mathbb{F} = \{([z_1, z_2, z_3], [w_1, w_2, w_3]) \in \mathbb{CP}_2 \times \mathbb{CP}_2 | z \cdot w = 0\} .$$

Which smooth compact four-manifolds M admit half-conformally-flat metrics? Certainly not all, for example, neither $S^2 \times S^2$ nor $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ admit such metrics, since these manifolds have signature zero, which would force any putative half-conformally-flat metric to be conformally-flat — whereas [17] the only simply-connected conformally-flat manifold is S^4! There is a reasonable hope, however, that, for any M, its connected sum $M \# m\mathbb{CP}_2$, $m \gg 0$, with a sufficiently large number of complex projective planes might always admit such metrics. (Note added in proof: This has now been proved by C. H. Taubes, J. Differential Geom. **36** (1992), 163–253.) While this remains beyond the scope of current technology, a panoply of interlocking methods [**26**, **6**, **7**, **18**, **19**, **20**] has evolved in the past several years for the construction of self-dual metrics on connected sums of self-dual manifolds. The present article will focus on a particularly elementary and explicit construction of this type, but which is limited to the case of metrics with an isometric S^1-action.

2. Constructing self-dual manifolds

Suppose that (M_{free}, g) is a self-dual four-manifold with a free isometric circle action generated by a Killing field ξ, and let $\xi^b = g(\xi, \cdot)$ be the corresponding one-form. We may then equip the three-manifold $X := M/S^1$ with the unique metric h for which the canonical projection π becomes a Riemannian submersion; i.e. such that

$$\pi^* h = \left(g - \frac{\xi^b \otimes \xi^b}{\|\xi\|^2} \right) .$$

Let β be the unique one-form on X such that

$$\pi^* \beta = \frac{-d\|\xi\|^2 + 2 * \xi^b \wedge d\xi^b}{2\|\xi\|^2} ,$$

and define a connection $\mathbf{D}$ on X by

$$\mathbf{D}_v w := \nabla_v w + \beta(v)w + \beta(w)v - g(v, w)\beta^{\#},$$

where $\beta = h(\beta^{\#}, \cdot)$ and ∇ is the Riemannian connection of h. If we replace g by αg, where α is any S^1-invariant function on M, then the connection $\mathbf{D}$ and the conformal class $[h]$ remain unchanged. By construction, the torsion-free connection $\mathbf{D}$ preserves the conformal structure $[h]$ in the sense that parallel transport preserves angles, and is thus a so-called *Weyl connection*

$$\mathbf{D}h = \omega \otimes h \ , \tag{2.1}$$

where $\omega = -2\beta$. The hypothesis that (M, g) is self-dual then has the consequence that the symmetrization of the Ricci tensor $\mathbf{r}_{\mathbf{D}}$ of $\mathbf{D}$ is a multiple of h; i.e. there is a function $\lambda : X \to \mathbb{R}$ such that

$$\mathbf{r}_{\mathbf{D}}(v, w) + \mathbf{r}_{\mathbf{D}}(w, v) = \lambda h(v, w) \quad \forall v, w . \tag{2.2}$$

We will call a three-manifold X equipped with a conformal metric h and a connection $\mathbf{D}$ satisfying (2.1) and (2.2) an *Einstein-Weyl* manifold. Such geometries were first studied by Elie Cartan [4], but their relation to self-dual four-manifolds was first recognized by Hitchin [12]. Several critical further observations described below were then made by Jones and Tod [14].

In order to reconstruct a self-dual four-manifold from an Einstein-Weyl geometry, we need an extra piece of information, namely the function $V = \|\xi\|^{-1}$. If we think of $M \to X$ as a circle bundle, we may equip it with a connection θ whose horizontal spaces are the g-orthogonal complements of the fibers. The self-duality of g then implies that the curvature of θ is given by $d\theta = *(d - \beta)V$.

We may invert this construction as follows: let $(X, [h], \mathbf{D})$ be an Einstein-Weyl three-manifold, and let $V : X \to \mathbb{R}$ be a positive solution of the elliptic equation $d * (d - \beta)V = 0$. Assume, in addition, that the closed 2-form $\frac{1}{2\pi}*(d-\beta)V$ represents an integral class in the de Rham cohomology $H^2(X)$. Then, by the Chern-Weil theorem, there is a circle-bundle $\pi : M \to X$ which admits a connection θ whose curvature is $d\theta = *(d - \beta)V$. Then, for any positive function μ on M, the metric $g = \mu(\pi^* h + V^{-2}\theta \otimes \theta)$ is self-dual. Often one takes $\mu = V$, so that the above expression becomes $g = \pi^* Vh + V^{-1}\theta \otimes \theta$.

EXAMPLE 2.3. Take X to be $\mathbb{R}^3$ punctured at n points $p_1, \ldots, p_n$, with h the Euclidean metric and $\mathbf{D}$ the usual flat connection, and let V be the sum of their Green's functions:

$$V = \sum_{j=1}^{n} \frac{1}{2r_j} ,$$

where r_j is the Euclidean distance from p_j. Then $\frac{1}{2\pi} * dV$ has integral -1 on a small sphere around any one of the puncture points p_j; since such spheres generate $H_2(\mathbb{R}^3 - \{p_1, \ldots, p_n\})$, we conclude that $[\frac{1}{2\pi} * dV]$ is an integral de Rham class. We can therefore consider the circle bundle $M_{free} \to (\mathbb{R}^3 - \{p_1, \ldots, p_n\})$ with connection one-form θ whose curvature is $*dV$. There is then a self-dual metric on M given by $g = Vh + V^{-1}\theta \otimes \theta$. This is the metric of Gibbons and Hawking [9]. If we add n points $\hat{p}_1, \ldots, \hat{p}_n$ to M_{free} to obtain a new space M which comes equipped with a circle action having $\hat{p}_1, \ldots, \hat{p}_n$ as its fixed points and a projection $M \to \mathbb{R}^3$, then M admits a unique smooth structure such that g extends to M as a smooth Riemannian metric:

$$\begin{array}{ccccc} M & = & M_{free} & \cup & \{\hat{p}_1, \ldots, \hat{p}_n\} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}^3 & = & (\mathbb{R}^3 - \{p_1, \ldots, p_n\}) & \cup & \{p_1, \ldots, p_n\} \end{array}$$

Moreover, the resulting Riemannian manifold is complete and, by virtue of special properties of the Einstein-Weyl space $\mathbb{R}^3$, actually Ricci-flat Kähler. With a little care, this construction can easily be generalized to the case of infinitely many (sparsely located) centers [1].

EXAMPLE 2.4. [18]. Let $(X, h, \mathbf{D})$ be hyperbolic 3-space $\mathscr{H}^3$ punctured at n points $p_1, \ldots, p_n$, where $\mathbf{D}$ is the Riemannian connection. We again build V from the Green's functions of the given points

$$V = 1 + \sum_{j=1}^{n} \frac{1}{e^{2\rho_j} - 1}, \tag{2.5}$$

where ρ_j denotes the hyperbolic distance from p_j. Then $[\frac{1}{2\pi} * dV]$ is again an integral class, and we can define a circle bundle $M_{free} \to (\mathscr{H}^3 - \{p_1, \ldots, p_n\})$ with connection one-form θ whose curvature is $*dV$. Let ρ denote the hyperbolic distance from any reference point. The metric

$$g = (\operatorname{sech}^2 \rho)\,(\pi * Vh + V^{-1}\theta \otimes \theta) \tag{2.6}$$

is then self-dual, and, because of our choice of conformal gauge, may be smoothly compactified by adding a two-sphere and n points $\hat{p}_1, \ldots, \hat{p}_n$. Indeed, let B denote the closed unit ball in $\mathbb{R}^3$, and identify the interior of B with $\mathscr{H}^3$ via the Poincaré conformal model. Then $M = M_{free} \cup S^2 \cup \{p_1, \ldots, p_n\}$ can be made into a smooth four-manifold with circle-action in such a manner that $S^2 \cup \{p_1, \ldots, p_n\}$ is the fixed point set and B is the orbit space, so that the projection to B is as follows:

$$\begin{array}{ccccccc} M & = & M_{free} & \cup & S^2 & \cup & \{\hat{p}_1, \ldots, \hat{p}_n\} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B & = & (\mathscr{H}^3 - \{p_1, \ldots, p_n\}) & \cup & \partial B & \cup & \{p_1, \ldots, p_n\} \end{array}$$

Calculations similar to those involved in the analysis of Example (2.3) then show

THEOREM 2.7. *The metric* g *of equation* (2.6) *has nonnegative scalar curvature, and extends to* M *to yield a compact self-dual four-manifold diffeomorphic to the* n*-fold connected sum* $\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2$.

When $n = 0, 1$, this construction produces the standard metrics on S^4 and $\mathbb{CP}_2$, respectively. When $n = 2$, we instead get the self-dual metrics on $\mathbb{CP}_2 \# \mathbb{CP}_2$ first found in [**26**].

EXAMPLE 2.8 [**19, 20, 15**]. Let Y be a hyperbolic three-manifold with smooth conformal compactification $\overline{Y}$, meaning that we assume that $\overline{Y}$ is a smooth compact three-manifold-with-boundary such that $Y = \overline{Y} - \partial\overline{Y}$, and the hyperbolic metric h of Y is of the form $h = f^{-2}\hat{h}$ for $\hat{h}$ a smooth Riemannian metric on $\overline{Y}$ and f a nondegenerate defining function of the boundary $\partial\overline{Y}$; by Thurston's main theorem [**29**], the class of $\overline{Y}$ admitting structures of this kind includes "most" atoroidal three-manifolds-with-boundary. Let $p_1, \ldots, p_n \in Y$ be given, let G_j be the corresponding Green's functions, and set $X = Y - \{p_1, \ldots, p_n\}$. Then we can mimic the previous construction of compact self-dual four-manifolds by taking

$$V = 1 + \sum_{j=1}^{n} G_j ,$$

trying to find a circle-bundle with connection one-form θ whose curvature is $*dV$, setting

$$g = f^2\ (\pi^* Vh + V^{-1}\theta \otimes \theta) ,$$

and compactifying by adding a copy of $\partial\overline{Y}$ and n isolated fixed points $\{\hat{p}_1, \ldots, \hat{p}_n\}$. The only catch lies in showing that $[\frac{1}{2\pi} * dV]$ is an integral cohomology class and, indeed, this will usually only be true for some special configurations of points! Nonetheless, one can verify the integrality condition in many cases. For example, if Y is a handle-body, the integrality condition is automatically verified, and one may use this to construct explicit self-dual metrics on arbitrary connected sums of $S^1 \times S^3$'s and $\mathbb{CP}_2$'s. On the other hand, if $Y = S_{\mathbf{g}} \times \mathbb{R}$, where $S_{\mathbf{g}}$ is a compact surface of genus $\mathbf{g} \geq 2$, one finds that the integrability condition is nontrivial, but, by restricting ones choice of point-configurations, the construction can be made to yield self-dual metrics on $(S^2 \times S_{\mathbf{g}}) \# n\mathbb{CP}_2$ provided that $n \neq 1$.

3. Twistor spaces

All the self-dual manifolds described in the previous section of course are associated with complex three-manifolds, namely their twistor spaces, and these complex manifolds completely encode the conformal geometry of each self-dual four-manifold, providing a higher-dimensional analog of the familiar dictionary between Riemann surfaces and complex curves. The fact that the metrics in question have conformal Killing fields is then reflected by a $\mathbb{C}^*$-action on their twistor spaces. At least locally, one can then construct the quotient of the twistor space by this action, thereby producing a complex surface, called the *minitwistor space* **[14]**, which corresponds to the Einstein-Weyl quotient geometry **[12]**. Let us now examine our key examples in this light. We begin with the Gibbons-Hawking metrics of Example (2.3), the twistor spaces of which were discovered by Hitchin **[10]**. The relevant Einstein-Weyl geometry is in this case that of Euclidean three-space, and the corresponding mini-twistor space **[13]** is $T\mathbb{CP}_1$. Let $\mathcal{O}(k) \to T\mathbb{CP}_1$ denote the pull-back of the degree k line-bundle over $\mathbb{CP}_1$ via the canonical projection. The data points $p_1, \dots, p_n \in \mathbb{R}^3$ specify n sections of $T\mathbb{CP}_1 \to \mathbb{CP}_1$, and these are the zero loci of n sections $P_1, \dots, P_n$ of $\mathcal{O}(2)$. In the total space of the rank 2 vector bundle $\mathcal{O}(n)\oplus\mathcal{O}(n)$, let $\tilde{Z}$ denote the hypersurface $xy = \prod_{j=1}^n P_j$, where x and y refer to the two factors of $\mathcal{O}(n)\oplus\mathcal{O}(n)$. The twistor space Z of the Gibbons-Hawking metric is then given by a "small resolution" of this three-dimensional complex algebraic variety, meaning that each singular point is replaced by a rational curve. For an important generalization of this class of twistor spaces, see **[16]**.

We now turn to the manifolds given by Example (2.4). In this case, the relevant Einstein-Weyl geometry is that of hyperbolic three-space, and the corresponding minitwistor space is $\mathbb{CP}_1 \times \mathbb{CP}_1$. Let $\mathcal{O}(k, l)$ denote the unique holomorphic line-bundle over $\mathbb{CP}_1 \times \mathbb{CP}_1$ with degree k on the first factor and degree l on the second, and let the data points $p_1, \dots, p_n \in \mathcal{H}^3$ correspond to the zero loci of n sections $P_1, \dots, P_n \in \Gamma(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}(1, 1))$. Let $\mathcal{B}$ denote the total space of the $\mathbb{CP}_2$-bundle

$$\mathcal{B} := \mathbb{CP}(\mathcal{O}(n-1, 1) \oplus \mathcal{O}(1, n-1) \oplus \mathcal{O}) \xrightarrow{\pi} \mathbb{CP}_1 \times \mathbb{CP}_1 , \tag{3.1}$$

and define an algebraic variety $\tilde{Z} \subset \mathcal{B}$ by the equation

$$xy = t^2 \prod_{j=1}^{n} P_j , \tag{3.2}$$

where $x \in \mathcal{O}(n-1, 1)$, $y \in \mathcal{O}(1, n-1)$, and $t \in \mathcal{O} := \mathcal{O}(0, 0)$. The twistor space Z of the metric constructed in Example (2.4) is then obtained from $\tilde{Z}$ by making small resolutions of the singular points and blowing down the surfaces $x = t = 0$ and $y = t = 0$ to $\mathbb{CP}_1$'s. Notice that Hitchin's twistor spaces are degenerations of these.

These twistor spaces thus turn out to be *Moishezon spaces*, meaning that they are bimeromorphic to smooth projective varieties; for example, when $n = 2$, the above twistor space is bimeromorphic to a resolution of the intersection of two hyper-quadrics in $\mathbb{CP}_5$ with four ordinary double points **[26]**. For $n > 3$ one can also show **[5, 22]** that their generic small deformations are not even bimeromorphic to *Kähler* manifolds, so that one observes from these twistor examples a rather unexpected phenomenon of broader interest: the class of compact complex manifolds bimeromorphic to Kähler is not stable under deformation of complex structure.

4. Scalar-flat Kähler surfaces with symmetry

As mentioned in §1, an interesting class of half-conformally-flat four-manifolds is given by the scalar-flat Kähler surfaces, i.e. complex two-manifolds with Kähler metrics with scalar curvature $\equiv 0$. A beautiful characterization of such metrics in terms of their twistor spaces was found by Pontecorvo **[25]**. Namely, the complex structure J, as well as the conjugate complex structure $-J$, are, by definition, sections of the twistor fibration. These are "conjugate," in the sense that they are interchanged by the antipodal map on each fiber of the twistor fibration. (This fiber-wise antipodal map is an anti-holomorphic involution of the twistor space, and will henceforth be called the "real structure.") The integrability condition on J then implies that the image of each of these two sections are complex hypersurfaces, say Σ and $\overline{\Sigma}$, in the twistor space Z. The fact that the metric is Kähler then implies that the line bundle $(\Sigma\overline{\Sigma})^2$ is isomorphic to the anti-canonical bundle K^{-1} of the twistor space Z; the crux of the argument is that the Kähler form, being parallel, corresponds by the Penrose transform **[11]** to a holomorphic section of $K^{-1/2}$, and the zero-locus of this section is, by inspection, $\Sigma \coprod \overline{\Sigma}$. Conversely, if a holomorphic section of $K^{-1/2}$ is invariant under the real structure of a twistor space, it defines a scalar-flat Kähler metric in the conformal class on the open subset of the four-manifold over which the given holomorphic section has two distinct zeroes on each twistor fiber.

A remarkable consequence of this is the following: a self-dual manifold arising as in Example (2.3) or (2.4) is automatically conformally isometric to a number of different scalar-flat Kähler surfaces! Indeed, as we saw in §3, the corresponding twistor spaces Z admit $\mathbb{C}^*$-actions for which the stable quotient is either $T\mathbb{CP}_1$ or $\mathbb{CP}_1 \times \mathbb{CP}_1$. But this means that a section of $K^{-1/2}$ on $T\mathbb{CP}_1$ or $\mathbb{CP}_1 \times \mathbb{CP}_1$, respectively, will pull back as section of $K^{-1/2}$ on Z. Since these complex surfaces admit many half-anti-canonical divisors, the claim follows. For applications of these ideas to the construction of *compact* scalar-flat Kähler surfaces via Example (2.5), see **[20]**.

Let us now focus on the twistor spaces (3.2) arising from Example (2.4), in which there are two distinguished families of effective divisors, corresponding to the two factors of the projection $\pi : \tilde{Z} \to \mathbb{CP}_1 \times \mathbb{CP}_1$ in (3.1). These two families, henceforth denoted by $|D|$ and $|\overline{D}|$, are interchanged by the real structure. The intersection of any such divisor D with its conjugate $\overline{D}$ is just a fiber of the projection π; moreover, it is a *real twistor line*, corresponding to the twistor fiber of a point of the fixed two-sphere of the circle action on $n\mathbb{CP}_2$. When this twistor line is removed, the resulting space N is the blow-up of $\mathbb{C}^2$ at n-points along a complex line in $\mathbb{C}^2$. When the conformal factor $(\operatorname{sech}^2 \rho)$ in (2.6) is replaced by e^{2B}, where $B : \mathscr{H}^3 \to \mathbb{R}$ is the

Busemann function of any geodesic, the resulting metric

$$e^{2B}(\pi^* Vh + V^{-1}\theta \otimes \theta), \tag{4.1}$$

where V is given by (2.5), is a scalar-flat Kähler metric on N. (Of course, the orientation on N is opposite that of M.) Moreover, the S^1-action is carried over to N to a conformal action. It turns out that these properties completely characterize our metrics:

THEOREM 4.2. *Suppose that N is a complete scalar-flat Kähler surface. If the metric is asymptotically Euclidean and admits a nontrivial conformal Killing field, it is isometric to* (4.1).

PROOF. When N is asymptotically Euclidean, it can be conformally compactified by adding one point at infinity, henceforth called o. After a change of orientation, the conformal curvature of the resulting compact conformal Riemannian manifold M is self-dual. Let $\wp : Z \to M$, be the twistor fibration. Thus Z is a compact complex manifold.

Over $M\backslash\{o\}$, the twistor fibration has two sections, namely the complex structure J on N and its conjugate $-J$. The integrability of J implies that the images Σ and $\overline{\Sigma}$ of these sections are complex submanifolds of Z. Let D and $\overline{D}$ respectively denote the closures of Σ and $\overline{\Sigma}$. Since $D\backslash\Sigma$ is contained in the twistor line $L_o = \wp^{-1}(o)$ of the point at infinity, $D \cup L_o$ is $*$-analytic in the sense of Mumford **[23]**, and hence a subvariety of Z. Thus $D \subset Z$ is a complex hypersurface. The same applies to $\overline{D}$.

We already know that the divisor D is nonsingular away from the twistor line L_o. On the other hand, a generic twistor fiber intersects D transversely in a single point, so the homological intersection number $D \cdot L_o$ is 1. Let $z \in Z$ be any point. Through z passes a two-complex-parameter family of $\mathbb{CP}_1$'s with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ and representing the fiber homology class. The tangent spaces of these curves fill out an open cone in $T_z Z$. In particular, through every point of $z \in D$ passes such a $\mathbb{CP}_1$ meeting D in precisely one point. Fix another point z' on such a curve, and consider the two-parameter family of nearby $\mathbb{CP}_1$'s through z'. Each such curve meets D in one point. This provides a holomorphic chart for a neighborhood of $z \in D$. Hence the divisor D is a nonsingular surface.

Let us consider the divisor $\Sigma + \overline{\Sigma}$ in the twistor space $Z\backslash L_o$ of our original noncompact complex surface. By Pontecorvo's theorem **[25]**, the fact that the metric is scalar-flat Kähler implies that there is a line bundle isomorphism $\Sigma\overline{\Sigma} \cong \mathbf{K}^{-\frac{1}{2}}$. Let $\phi \in \Gamma(Z\backslash L_o, (\mathbf{D}\overline{\mathbf{D}})^* \otimes \mathbf{K}^{-\frac{1}{2}})$ be the section which realizes this isomorphism. Since $L_o \subset Z$ has complex codimension 2, this section extends across L_o by Hartog's theorem. But the extended section is nonzero away from the codimension 2 set L_o, and hence is everywhere nonzero. This shows that

$$\mathbf{D}\overline{\mathbf{D}} \cong \mathbf{K}^{-\frac{1}{2}}. \tag{4.3}$$

After this observation, we can show that the surface D contains L_o. If not, both D and $\overline{D}$ intersect L_o transversely in one point each, and these points are interchanged by Σ. Thus $D \cap \overline{D} = \varnothing$, and we can apply Pontecorvo's theorem **[25]** to conclude that there is a scalar-flat Kähler metric on M in the fixed conformal class. This metric and our original Kähler metric g on $N = M \backslash \{o\}$ are thus conformally related and Kähler with respect to the same complex structure J. Since the complex dimension of N is > 1, the conformal factor is therefore constant. This would then imply that (M, g) is incomplete, which is a contradiction.

To finish our proof, we will need the following

PROPOSITION 2.5. *The surface D is isomorphic to $\mathbb{CP}_2$ blown up at a collinear collection of points.*

PROOF. With the isomorphism (4.3), we apply the adjunction formula on D to find the canonical class on D in terms of the restriction of the divisor classes of D and $\overline{D}$. Then applying the same formula to L_o as a divisor on D, we find that the self-intersection of this real twistor line is equal to one. It follows that D is a rational surface (see e.g. **[3**, Proposition V.4.3**]**). In particular, $h^1(D, \mathcal{O}) = 0$. With the last equality and the following exact sequence on D: $0 \to \mathcal{O} \to \mathcal{O}(\mathrm{L}_o) \to \mathcal{O}_{L_o}(1) \to 0$, we find that the twistor line L_o is in a net of rational curves. The associated map of this complete linear system exhibits the divisor D as the blow up of $\mathbb{CP}_2$.

Now the naturality of the twistor construction insures that any conformal automorphism of a self-dual manifold M induces a holomorphic automorphism of its twistor space. A conformal Killing field therefore lifts to a holomorphic vector field on Z. If this holomophic vector field is not tangent to D, we conclude that $\dim|D| \geq 1$; similarly for $|\overline{D}|$. In this case, we may then apply (3.1) in **[27]** to conclude our result. Otherwise, our one-parametergroup of conformal transformation of M induces holomorphic transformations of Z preserving on D.

Since a holomorphic transformation homotopic to the identity must leave any exceptional divisor fixed, the transformation of D descends to be a transformation of $\mathbb{CP}_2$ leaving the blown-up points fixed. As the twistor line at infinity is also invariant, there is an additional pair of fixed points, say, p and q. This is possible only if all these fixed points lie on two lines. As the line joining p and q does not contain any points of blowing up, all the blown-up points are collinear. □

When all the blown-up points are collinear, the system $|D|$ on the surface D contains at least one member, namely the proper transform of the line containing all the blown up points. This is so because the isomorphism (4.3) implies that the divisor class of D is linearly equivalent to the proper transform of the line passing through all points of blowing up.

The restriction of the twistor fibration to D induces an isomorphism of first homotopy groups between D and M, since this map is a diffeomorphism away from the twistor line at infinity, which it contracts to a point. Thus M is simply-connected. But, by the Penrose transform, we therefore have $h^1(Z, \mathscr{O}) = b_1(M) = 0$ [**11**]. The exact sequence $0 \to \mathscr{O} \to \mathscr{D} \to \mathscr{O}_D(\mathbf{D}) \to 0$, on the twistor space now tells us that the linear system $|D|$ on Z has dimension one. According to theorem (3.1) in [**27**], the metric on M is therefore conformal to the metric (4.1). Moreover, the two complex structures agree. Since two Kähler metrics in the same conformal class on a connected complex surface can only differ by a homothety, the proof of our main theorem is finished. □

REMARK 4.4. The first part of the above proof can easily be adapted to prove the following more general result: *Let M be an asymptotically Euclidean, scalar-flat Kähler surface, perhaps with many ends. Then M is biholomorphic to $\mathbb{C}^2$ blown up at a finite number of points.* In particular, such a manifold can only have one end, and is automatically simply connected. The reason for the "one end" conclusion is that any real twistor line "at infinity" will have self-intersection 1 in the surface D; yet two distinct real twistor lines must be disjoint! Surface classification thus excludes the possibility of the surface D containing two real twistor lines.

REMARK 4.5. If M is just anti-self-dual Hermitian instead of scalar-flat Kähler, theorem (4.2) will still hold provided one imposes the additional hypothesis that M be simply connected. Indeed, without any assumptions whatsoever, the isomorphism (4.3) generalizes to become $\mathbf{D}\overline{\mathbf{D}} \cong \mathbf{K}^{-\frac{1}{2}}\mathbf{F}$ where $\mathbf{F}$ is a holomorphic line bundle with vanishing Chern class. The assumption of simple connectivity then forces the torsion bundle $\mathbf{F}$ to be trivial. One can then once again apply Pontecorvo's theorem. However, scalar-flat anti-self-dual Hermitian counter-examples with fundamental group $\mathbb{Z}$ may be easily constructed by removing a point from a Hopf surface $S^3 \times S^1$.

REMARK 4.6. The Kähler metric (4.1) was originally found [**18**] by the method of Kähler reduction. The present theorem of course shows that this (or any other) method will not lead to other asymptotically Euclidean solutions. Nevertheless, there are many other complete Kähler surfaces of constant scalar curvature which may be found by this approach [**20**] and its generalizations [**28**].

REMARK 4.7. The metrics (4.1) can of course be conformally compactified so as to yield self-dual metrics on connected-sums of complex projective planes, and the corresponding compact four-manifolds automatically admit an isometric circle-action. An interesting generalization of the problem solved in this section would thus be that of classifying compact self-dual four-manifolds with nontrivial isometric S^1-action. The first author has recently succeeded in proving that, provided the action is assumed to be **semifree**, any such manifold with nonnegative scalar curvature must either arise as in Example (2.4), or else be conformally flat; however, it also seems certain that there will exist classes of examples for which the semifree hypothesis will fail.

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