Duality and Yang–Mills fields on quaternionic Kähler manifolds

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The concept of a self dual connection on a four-dimensional Riemannian manifold is generalized to the 4n-dimensional case of any quaternionic Kähler manifold. The generalized self-dual connections are minima of a modified Yang–Mills functional. It is shown that our definitions give a correct framework for a mapping theory of quaternionic Kähler manifolds. The mapping theory is closely related to the construction of Yang–Mills fields on such manifolds. Some monopole-like equations are discussed.

I. INTRODUCTION

A quaternionic Kähler manifold is a Riemannian manifold whose holonomy group can be reduced to a subgroup of Sp(n) · Sp(1), n > 1.\textsuperscript{1,2} By definition, such manifold has dimension 4n. As demonstrated by Salamon,\textsuperscript{2,3} it can be also viewed as a higher-dimensional analog of the anti-self-dual Einstein four-manifold. The bundle of two-forms on a quaternionic Kähler manifold \( M \) has the following irreducible decomposition as representation of Sp(n) · Sp(1):

\[
\Lambda^2 T^*M = S^2H \oplus S^2E \oplus (S^2H \oplus S^2E)^1,
\]

(1.1)

where \( H \) and \( E \) are vector bundles associated to the standard representations of Sp(n) and Sp(1), respectively. This decomposition resembles the decomposition of \( \Lambda^2 T^*M \) into the direct sum of self-dual and anti-self-dual two-forms when \( M \) is four dimensional. Just as in the four-dimensional case we are able to interpret the decomposition (1.1) in terms of the Hodge *-operator.

If the curvature of a connection \( \nabla \) is in either the \( S^2H \) or the \( S^2E \) part of (1.1) then \( \nabla \) is a minimum of the Yang–Mills functional and if the curvature is in the orthogonal complement of \( S^2H \oplus S^2E \) then \( \nabla \) is most likely a saddle point. We have found that the Yang–Mills functional can be modified so that whenever the curvature of \( \nabla \) is in one and only one component of (1.1) the connection is its minimum.

We demonstrate that our definitions are compatible with the description of Yang–Mills fields on four-manifolds and that they give a correct framework for mapping theory of quaternionic Kähler manifolds. On the other hand, when the energy functional is interpreted as a classical Lagrangian, our quaternionic mapping theory yields many new examples of quantum field theories with SU(2) [or SO(3)] gauge symmetry and composite gauge fields: four-dimensional sigma models. We show that some fundamental properties of the well-known four-dimensional \( \sigma \)-models on the quaternionic projective spaces are shared by such models on arbitrary quaternionic Kähler manifolds. Finally, we demonstrat that our formalism provides a global picture for the generalized monopole equation of Pedersen and Poon.\textsuperscript{4}

II. DUALITY

Let \( M \) be a 4n-dimensional Riemannian manifold whose holonomy group is contained in Sp(n) · Sp(1) \( \subset \) SO(4n). Then the cotangent bundle of \( M \) can be identified with

\[
T^*M - E \oplus \mathbb{H},
\]

where \( E \) and \( \mathbb{H} \) are the standard representations of Sp(n) and Sp(1), respectively. Then \( S^2\mathbb{H} \) is a real rank 3 subbundle of End \( TM \). Locally, at each \( x \in M \), \( S^2\mathbb{H} \) has a basis \( \{I, J, K\} \) satisfying

\[
I^2 = J^2 = -1, \quad NI = -JI = K.
\]

(2.1)

The metric \( g \) on \( M \) is compatible with the bundle \( S^2\mathbb{H} \) in the sense that for each \( A \in S^2\mathbb{H} \), \( g \) is Hermitian with respect to \( A \), i.e.,

\[
g(AX,AY) = g(X,Y) \quad \text{for all } X, Y \in T_xM.
\]

One can use the metric to define an isomorphism

\[
End TM \cong I^*M \oplus I^*M
\]

under which \( S^2\mathbb{H} \) is isometrically embedded in \( \Lambda^2 T^*M \). Explicitly, any element \( A \in S^2\mathbb{H} \) is mapped into \( \omega_A \) by

\[
\omega_A(X,Y) = g(AX,Y) \quad X, Y \in T_xM.
\]

Let \( \{\omega_1, \omega_2, \omega_3\} \) be a local orthogonal frame of \( S^2\mathbb{H} \subset \Lambda^2 T^*M \). For convenience of further computations let us normalize \( \{\omega_1, \omega_2, \omega_3\} \) to have length 2n and then define

\[
\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3.
\]

(2.2)

This \( \Omega \) is a globally defined, nondegenerate four-form on \( M \) and it is parallel. It is usually called the fundamental four-form or the quaternionic structure on \( M \). If \( \nabla \) is parallel then \( \nabla \Omega = 0 \) can be used to define quaternionic Kähler geometry in dimension bigger than 4. In dimension 4 we shall say that \( M \) is quaternionic Kähler if it is self-dual and Einstein. The parallelism of \( \Omega \) immediately implies that \( d\Omega = 0 \). Recently, Swann\textsuperscript{5} showed that the converse is also true provided \( \dim M \geq 12 \).

Pointwisely, \( \Omega \) can be described as follows. At any point \( x \in M \), \( T^*_xM = E_x \oplus H_x \), where \( E_x \) is the 2n-dimensional complex representation of Sp(n) and \( H_x \) is the two-dimensional complex representation of Sp(1). Let \( \omega_E \) and \( \omega_H \) be the sympletic forms on \( E_x \) and \( H_x \), respectively, and \( j_E \) and \( j \) the quaternionic structures. Then the metric \( g \) on \( T^*_xM \) can be expressed as

\[
\text{[MathJax formula]}.
\]
Let \( \{e^i, je^j; j = 1, \ldots, n\} \) be a symplectic basis on \( E \), and \( \{h_jh_j\} \) a symplectic basis on \( H \). We define

\[
\omega_0 = \sqrt{2} (e^i \otimes h + je^j \otimes h_j),
\omega_1 = (1/2) (e^i \otimes h - je^j \otimes h_j),
\omega_2 = (1/2) (je^j \otimes h - e^i \otimes h_j),
\omega_3 = (1/2) (j e^j \otimes h + e^i \otimes h_j).
\]

(2.3)

Now \( \{\omega_0, \omega_1, \omega_2, \omega_3; j = 1, \ldots, n\} \) forms an orthonormal basis on \( T^*M \). Let

\[
\omega_1 = \sum_{i,j} \omega_0^i \wedge \omega_1^j + \omega_2^i \wedge \omega_3^j,
\omega_2 = \sum_{i,j} \omega_0^i \wedge \omega_2^j + \omega_1^i \wedge \omega_3^j,
\omega_3 = \sum_{i,j} \omega_0^i \wedge \omega_3^j + \omega_1^i \wedge \omega_2^j.
\]

(2.4)

Then \( \{\omega_1, \omega_2, \omega_3\} \) forms an orthogonal basis on \( S^2H \). We shall choose \( \Omega \) as in (2.2). The orthogonal basis for \( S^2E \) can be written as

\[
\Sigma_{ij}^n = (\omega_0^i \wedge \omega_0^j + \omega_1^i \wedge \omega_1^j + \omega_2^i \wedge \omega_2^j + \omega_3^i \wedge \omega_3^j),
\]

for all \( 1 \leq i < j \leq n \).

(2.5)

Here, \( \Sigma_{ij}^n \) give \( n(n-1)/2 \) basis elements and \( \Sigma_{ij}^n, A = 1,2,3 \), give \( n(n+1)/2 \) basis elements, respectively. One can easily check that

\[
\text{vol}(M) = \frac{1}{2n+1}! \Omega^n
\]

(2.7)

\[
\text{vol}(M) = \frac{1}{2n+1}! \Omega \wedge \Omega^*.
\]

(2.8)

where \( \text{vol}(M) \) is the volume form of \( M \) and \( \Omega \) the Hodge \( \ast \)-operator. As a consequence we have

\[
\ast \Omega = \frac{6}{2n-1}! \Omega^*.
\]

(2.9)

Note that all these equations are valid even when \( n \) is equal to 1.

Definition 2.1: A two-form \( \omega \) on \( M \) is \( c \)-self-dual if

\[
\ast \omega = c \omega \wedge \Omega^{n-1}.
\]

(2.10)

When \( n = 1 \) then \( c = 1 \), because \( \ast \omega = \Omega \). The above equation is reduced to the conformally invariant self-dual or anti-self-dual equations on a four-dimensional oriented Riemannian manifold. Notice that the above definition depends on the choice of both the fundamental four form \( \Omega \) and the constant \( c \). In dimension higher than 4, as we shall now see, there are three different constants \( c \) that give nontrivial solutions to (2.10). Similar equations were studied in Ref. 6.

Theorem 2.2: Let \( \omega \) be a nonzero \( c \)-self-dual two-form. Then \( c = c_i, i = 1,2,3 \), where

\[
c_1 = \frac{6n}{(2n+1)!}, \quad c_2 = \frac{-1}{(2n-1)!}, \quad c_3 = \frac{3}{(2n-1)!}.
\]

(2.11)

Moreover, when \( c = c_1 \) then \( \omega \in S^2H \), when \( c = c_2 \) then \( \omega \in S^2E \), and when \( c = c_3 \) then \( \omega \) is in the orthogonal complement of \( S^2H \oplus S^2E \) in \( \Lambda^*T^*M \).

Proof: As the basis for \( S^2H \) is given in (2.5) and the basis for \( S^2E \) in (2.6) the proof is an easy exercise in linear algebra. Therefore, we only spell out the constraints on the coefficients of the two-form \( \omega \). Using the orthonormal basis \( \{\omega_0, \omega_1, \omega_2, \omega_3; j = 1, \ldots, n\} \) any two-form \( \omega \) can be written as

\[
\omega = \sum_{i,j} \omega_0^i \wedge \omega_1^j + \omega_2^i \wedge \omega_3^j.
\]

Then \( \ast \omega = c \omega \wedge \Omega^{n-1} \) if and only if

\[
\omega_0^i \wedge \omega_1^j + \omega_2^i \wedge \omega_3^j = 0, \quad \forall i,j.
\]

(2.13)

and

\[
\omega_0^i \wedge \omega_1^j = 0, \quad \forall i \neq j, \quad \forall \alpha, \beta.
\]

(2.14)

Similarly, \( \ast \omega = c_2 \omega \wedge \Omega^{n-1} \) if and only if

\[
\omega_0^i \wedge \omega_2^j = 0, \quad \forall i,j.
\]

(2.15)

Finally, \( \ast \omega = c_3 \omega \wedge \Omega^{n-1} \) if and only if

\[
\sum_{i=1}^n \omega_0^i = 0, \quad \forall i.
\]

(2.16)

Definition 2.3: Let \( P \) be a principal bundle on \( M \) with connection \( \nabla \). This connection is \( c \)-self-dual if its curvature two-form is \( c \)-self-dual.

Definition 2.4: For any real constant \( c \), a generalized 4D self-dual Yang–Mills functional on the space of connections on \( P \) is defined by
\[ \text{YM}_c(\nabla) \doteq \frac{1}{2} \int_M \left( \| F \|^2 + c^2 \| F \wedge \Omega^{-1} \|^2 \right) \text{vol}(M) , \]  

(2.17)

where \( F \) is the curvature of the connection.

\( \text{YM}_c(\nabla) \) has the following Euler-Lagrange equations

\[ d^*F + c^2 (d^* (F \wedge \Omega^{-1})) \wedge \Omega^{-1} = 0. \]  

(2.18)

Notice that

\[ 0 \leq \| F - c F \wedge \Omega^{-1} \|^2 = \| F \|^2 - 2 c (\text{tr} (F F \wedge \Omega^{-1}) + c^2 \| F \wedge \Omega^{-1} \|^2 \]  

\[ = \| F \|^2 - 2 c (\text{tr} (\Omega F \wedge F)) \wedge \Omega^{-1} + c^2 \| F \wedge \Omega^{-1} \|^2 \]  

\[ = \| F \|^2 - 16 c^2 \rho_1(P) \wedge \Omega^{-1} + c^2 \| F \wedge \Omega^{-1} \|^2 \]  

or

\[ c(8\pi^2) \rho_1(P) \wedge \Omega^{-1} \leq \| F \|^2 + c^2 \| F \wedge \Omega^{-1} \|^2 , \]

where \( \rho_1(P) \) is the first Pontrjagin class of the bundle \( P \) on \( M \). Hence, after integrating over \( M \), we get

\[ 8\pi^2 c \int_M \rho_1(P) \wedge \Omega^{-1} \text{vol}(M) \leq \text{YM}_c(\nabla) . \]  

(2.19)

The equality holds if and only if

\[ *F = c F \wedge \Omega^{-1} , \]

i.e., if \( F \) is \( c \)-self-dual. In such case we shall call the connection \( \nabla \) itself a \( c \)-self-dual connection. As \( \rho_1(P) \) is a topological invariant of the bundle \( P \), we define

\[ Q(P) \equiv 8\pi^2 \int_M \rho_1(P) \wedge \Omega^{-1} \text{vol}(M) \]  

(2.20)

and call it a topological charge of the bundle \( P \). We have just demonstrated the following proposition.

**Proposition 2.5:** Any \( c \)-self-dual connection is minimum of the Yang–Mills energy functional \( \text{YM}_c(\nabla) \).

The following result is due to Ref. 7.

**Proposition 2.6:** Any \( c \)-self-dual connection is an extremum of the Yang–Mills energy functional \( \text{YM}(\nabla) \). Moreover, \( c_1 \)– and \( c_1 \)-self-dual connections are minimizing.

**Proof:** Suppose \( \nabla \) is a \( c \)-self-dual connection. Then

\[ d^*F = c d^* (F \wedge \Omega^{-1}) = 0 \]

as \( dF = d\Omega = 0 \). Hence, \( d^*F = 0 \) or \( \nabla \) is a Yang–Mills connection.

Let us write \( F(\nabla) \in \Lambda^2 TM \) as

\[ F(\nabla) = F_1 + F_2 + F_3 , \]

where \( F_1 \in S^2 \mathbb{H} \), \( F_2 \in S^2 \mathbb{E} \), and \( F_3 \in (S^2 \mathbb{H} \otimes S^2 \mathbb{E}) \). Then

\[ \text{YM}(\nabla) \doteq \frac{1}{2} \int_M (\| F_1 \|^2 + \| F_2 \|^2 + \| F_3 \|^2) \text{vol}(M) \]

(1.1) is an orthogonal decomposition with respect to the usual norm \( \| \cdot \| \) on \( \Lambda^2 TM \). Notice that the topological charge of \( P \) can be written in terms of the components of \( F(\nabla) \):

\[ Q(P) = \int_M \text{tr} (F \wedge F) \wedge \Omega^{-1} \text{vol}(M) \]  

\[ = \int_M \left( \frac{1}{c_1} \| F_1 \|^2 + \frac{1}{c_2} \| F_2 \|^2 + \frac{1}{c_3} \| F_3 \|^2 \right) \text{vol}(M) . \]

Hence, we can write \( \text{YM}(\nabla) \) as

\[ 2\text{YM}(\nabla) = c_1 Q(P) + \int_M \left( \left( 1 - \frac{c_1}{c_3} \right) \| F_1 \|^2 + \left( 1 - \frac{c_2}{c_1} \right) \| F_2 \|^2 \right) \text{vol}(M) \]  

\[ + \left( 1 - \frac{1}{2n + 1} \right) \| F_3 \|^2 \text{vol}(M) , \]  

(2.21)

\[ 2\text{YM}(\nabla) = c_2 Q(P) + \int_M \left( \left( 1 - \frac{c_2}{c_3} \right) \| F_2 \|^2 + \left( 1 - \frac{1}{2n + 1} \right) \| F_3 \|^2 \right) \text{vol}(M) \]  

\[ + \left( 1 - \frac{1}{2n + 1} \right) \| F_1 \|^2 \text{vol}(M) , \]  

(2.22)

or

\[ 2\text{YM}(\nabla) = c_3 Q(P) + \int_M \left( \left( 1 - \frac{c_1}{c_2} \right) \| F_1 \|^2 + \left( 1 - \frac{1}{2n} \right) \| F_2 \|^2 \right) \text{vol}(M) \]  

\[ + \left( 1 - \frac{1}{2n} \right) \| F_3 \|^2 \text{vol}(M) . \]  

(2.23)

It follows now from (2.21), (2.22), and Theorem 2.2 that \( c_1 \)-and \( c_2 \)-self-dual connections are minima of \( \text{YM}(\nabla) \). We do not know of any examples of \( c_1 \)-self-dual connections but (2.23) seems to indicate that, if they exist, they will be unstable.

### III. QUATERNIONIC MAPS AND SIGMA MODELS

In this chapter we introduce a new concept of quaternionic maps. We shall do it in such a way that it generalizes the theory of holomorphic mappings between Kähler manifolds. On the other hand we shall see that it is also very natural in studying instantons on four-manifolds and four-dimensional \( \sigma \)-models with composite \( SU(2) \) or \( SO(3) \) gauge fields and Yang–Mills fields on quaternionic Kähler manifolds.

It is well-known that, if one defines a quaternionic Kähler submanifold to be a submanifold with a quaternionic structure given by restriction, then it is automatically a totally geodesic submanifold. We shall therefore not insist that the whole quaternionic structure be preserved by such mappings. Instead we adopt a weaker definition.

**Definition 3.1:** Let \( M, N \) be quaternionic Kähler manifolds. A map \( f \) from \( M \) to \( N \) is called quaternionic Kähler if \( f^* S^2 \mathbb{H}_N \subseteq S^2 \mathbb{H}_M \).

The following theorem is in an obvious analogy to the...
well-known result stating that holomorphic maps between Kähler manifolds are energy minimizing.

**Theorem 3.2:** On the space of differentiable mappings between two compact oriented quaternionic Kähler manifolds, \( M \) and \( N \), define the following functional:

\[
E(f) = \frac{1}{2} \int_M \left( \| f^*\omega_i \|^2 + c^2 \| f^*\omega_i \wedge \Omega^{-1} \|^2 \right) \text{vol}(M),
\]

where \( c = c_1 = 6m/(2m + 1) \), \( 4m = \dim M \), and

\[
Q(f) = \int_M f^*\Omega_N \wedge \Omega^{-1}. \tag{3.1}
\]

Then \( E(f) \geq Q(f) \) and the equality holds if and only if the map \( f \) is quaternionic.

**Proof:** Let \( \Omega_M, \Omega_N \) be the fundamental four-forms on \( M \) and \( N \), respectively. Once they are fixed, \( Q(f) \) is a homotopy invariant. As usual, we shall call it the degree or the topological charge of \( f \).

Let \( \{\omega_1, \omega_2, \omega_3\} \) be a local orthogonal frame on \( S^2\mathbb{H}_M \) such that

\[
\Omega_N = \omega_1 \wedge \omega_2 + \omega_2 \wedge \omega_3 + \omega_3 \wedge \omega_1.
\]

We have to show that \( E(f) \) is well defined. If \( \omega_i = \sum \phi_j \mu_j \) is an \( SO(3) \) rotation of the frame field on \( S^2\mathbb{H}_N \), then pointwise

\[
f^*\omega_i = \sum \phi_j f^*\mu_j.
\]

Furthermore,

\[
\sum_{i=1}^3 \| f^*\omega_i \|^2 = \sum_{i=1}^3 f^*\omega_i \wedge f^*\omega_i,
\]

\[
= \sum_{i=1}^3 \sum_{j,k} (\phi_j f^*\mu_j) \wedge (\phi_k f^*\mu_k)
= \sum_{i=1}^3 \sum_{j,k} (\phi_j \phi_k) (f^*\mu_j \wedge f^*\mu_k)
= \sum_{i=1}^3 \sum_{j,k} (\phi_j) (f^*\mu_j \wedge f^*\mu_k)
= \sum_{i=1}^3 \delta_{jk} f^*\mu_j \wedge f^*\mu_k = \sum_{i=1}^3 f^*\mu_j \wedge f^*\mu_j
= \sum_{i=1}^3 \| f^*\mu_j \|^2.
\]

Similarly,

\[
\sum_{i=1}^3 \| f^*\omega_i \wedge \Omega^{-1} \|^2
= \sum_{i=1}^3 (f^*\omega_i \wedge \Omega^{-1}) \wedge (f^*\omega_i \wedge \Omega^{-1})
= \sum_{i=1}^3 \sum_{j,k} (\phi_j \phi_k) (f^*\mu_j \wedge \Omega^{-1}) \wedge (f^*\mu_k \wedge \Omega^{-1})
= \sum_{j,k} (f^*\mu_j \wedge \Omega^{-1}) \wedge (f^*\mu_k \wedge \Omega^{-1})
= \sum_{i=1}^3 \| f^*\mu_i \wedge \Omega^{-1} \|^2.
\]

Hence, \( E(f) \) is independent of the choice of any normalized frame on \( S^2\mathbb{H}_N \) and therefore well defined. Now the inequality \( cQ(f) \leq E(f) \) follows from

\[
0 \leq \| f^*\omega_i - f^*\omega_i \wedge \Omega^{-1} \|^2
\]

which can be written as

\[
c(f^*\omega_i, f^*\omega_i \wedge \Omega^{-1}) < \| f^*\omega_i \|^2 + c^2 \| f^*\omega_i \wedge \Omega^{-1} \|^2).
\]

Since

\[
f^*\Omega_N = \sum_{i=1}^3 f^*\omega_i \wedge f^*\omega_i,
\]

the inequality \( cQ(f) \leq E(f) \) is simply obtained by summation of (3.3) over \( i \) and integration over \( M \).

Finally, when \( c = 6m/(2m + 1) \), the assertion that \( cQ(f) = E(f) \) is equivalent to the requirement that

\[
f^*\omega = cf^*\omega \wedge \Omega^{-1}
\]

holds for all \( \omega \in S^2\mathbb{H}_N \), or that \( f^*\omega \in S^2\mathbb{H}_M \) by Theorem 2.2., i.e., \( f \) is quaternionic.

**Example 3.3:** If \( \dim M = 4 \), \( S^2\mathbb{H} \cong \Lambda^2 \). As the Hodge *-operator is conformally invariant, any orientation preserving conformal automorphism is a quaternionic map in our sense.

In Ref. 8 Atiyah gave a geometric construction for all basic \( SU(2) \)-instantons, i.e., anti-self-dual Yang-Mills fields on the Euclidean four-sphere with topological charge \(-1\), as follows: The Euclidean four-sphere is viewed as the quaternionic projective line \( \mathbb{H}^1 \). The tautological bundle is the bundle \( \mathbb{H} \) with charge \(-1\). The natural connection \( \nabla \) of \( \mathbb{H} \) is anti-self-dual. Let \( f \) be an orientation preserving conformal automorphism which is not an isometry. Then \( f^*\nabla \), the pull-back connection of \( f^*\mathbb{H} \), is a new anti-self-dual connection.

**Example 3.4:** The above example can be easily generalized as follows: The quaternionic projective space \( \mathbb{H}P^n \) has a tautological bundle \( \mathbb{H} \). By definition, any element of \( GL^+(n+1,\mathbb{H}) \) is an orientation preserving quaternionic linear map. In other words, if \( f \in GL^+(n+1,\mathbb{H}) \) is considered as an automorphism of \( \mathbb{H}P^n \), then \( f^*\mathbb{H} \) is isomorphic to \( \mathbb{H} \). It follows that \( f^*\mathbb{H} = S^2\mathbb{H} \) and hence \( f \) is a quaternionic map. As the natural connection \( \nabla \) on \( \mathbb{H} \) is \( c_1 \)-self-dual, so is \( f^*\nabla \). Besides, as long as \( f^*\nabla \) is not isometry, \( f^*\nabla \) is not gauge equivalent to \( \nabla \). We do not know if these are all \( c_1 \)-self-dual connections on \( \mathbb{H}P^n \).

**Example 3.5:** Another well-known example of a mapping which in our language is quaternionic is a general \( SU(2) \)-instanton over four-sphere with the topological charge \( k \). The \( S^2\mathbb{H} \) bundle on the quaternionic projective space \( \mathbb{H}P^k \) has a canonical \( Sp(1) \)-connection and all instantons over \( S^4 \) are induced by an appropriate choice of \( f:S^4 \to \mathbb{H}P^k \). In fact \( f \) can be described explicitly as follows: If \( \phi \in \pi_1(\mathbb{H}P^k) \) is a local (Fubini-Study) quaternionic coordinate on the quaternionic projective space and \( \phi \in S^4 \) is a local quaternionic coordinate on the four-sphere identified with the quaternionic projective line \( \mathbb{H}P^1 \) then
\[ u(x) = [\lambda \cdot (B - x1)]^+, \]
where \( \lambda = (\lambda_1, ..., \lambda_k) \) is a quaternionic row vector, \( u \) is a quaternionic column vector, \( B \) is a symmetric quaternionic \( k \times k \) matrix, \( \dagger \) denotes quaternionic conjugation and transposition, and \( (\lambda, B) \) are subject to the following two conditions:

\[ \text{Im}(B^\dagger B + \lambda^\dagger \lambda) = 0, \]
\[ (\forall x \in \mathbb{H}^4 \cdot (B - x1) \xi = 0, \lambda \cdot \xi = 0 \text{ where } \xi \in \mathbb{H}^4 \Rightarrow \xi = 0. \]

In the same way \( k \)-instantons over the complex projective plane can be generated by quaternionic maps from \( \mathbb{C}P^2 \rightarrow \mathbb{P}^2 \).

The energy functional (3.1) may also be interpreted as an SO(3) locally gauge invariant Lagrangian of the interesting class of nonlinear field theories called \( \sigma \)-models. In particular, if \( \dim M = 4 \), one can think of \( M \) as a physical, possibly curved, space-time and \( f(x), x \in M \), becomes an \( N \)-valued classical field with the action functional given by \( E(f) \). \( E(f) \) is manifestly invariant with respect to the gauge coordinate transformation on \( M \) (diffeomorphisms of \( M \)) as well as it is gauge invariant under the following gauge transformations

\[ (f^*\omega_i) = \sum_j \Phi_0(x) (f^*\omega_j) \cdot x, \]

where \( \Phi_0(x) \) is a local SO(3) transformation and \( (f^*\omega_i) \) is the curvature two-form of a gauge field \( A \) on \( M \) defined as follows:

\[ d(f^*\omega_i) = \sum_{j,k} \epsilon_{ijk} A_j \wedge f^*\omega_k. \]

The gauge potential one-form on \( A \) transforms in the usual way

\[ \delta (\epsilon_{ijk} A) = - d_{A'} \Phi_q(x). \]

\( A', (f) \) depends on the choice of \( f(x), i.e., it is a composite gauge field. If \( N = \mathbb{H}^n \) and \( u \in \mathbb{H}^n \) as before then

\[ A(u) = - \frac{1}{2} u^d d^d u - d^d u^d u = iA_1 + jA_2 + kA_3. \]

This particular example was introduced and extensively studied by Gürsey and Tze. Here we see that many interesting global and local properties of \( \mathbb{H}^n \)-model are common for a large class of field theoretical models based on \( E(f) \). All of them have duality equations built in and all possess global topological invariants.

IV. GENERALIZED BOGOMOLNY EQUATIONS

In this section we discuss some special solutions of the \( c \)-self-duality equations. If \( M = \mathbb{R}^4 \otimes \mathbb{R}^n \) and \( P \) is a principal bundle over \( M \) then one can study \( x_0 \)-invariant solutions to the usual self-dual equations. They are called time invariant instantons or monopoles. In our case, let \( M = \mathbb{R}^4 \otimes \mathbb{R}^n \otimes \mathbb{C} \) be a principal bundle over \( M \) and let \( Y \) be our Yang–Mills functional. In an obvious analogy to the four-dimensional case we can study \( x_0 \)-invariant \( c \)-self-dual connections on \( M \) or \( "c-monopoles" \) on \( \mathbb{R}^4 \otimes \mathbb{R}^n \). Let us start with the following observation.

**Proposition 4.1:** Let \( M = \mathbb{R}^4 \otimes \mathbb{R}^n \) be the \( 4n \)-dimensional Euclidean flat space with global linear coordinates \( x_\alpha; \alpha = 0,1,2,3; i = 1,..., n \). For any \( (x_1, ..., x_n) \) in \( \mathbb{R}^n \) we define

\[ p: \mathbb{R}^4 \rightarrow \mathbb{R}^4 \otimes \mathbb{R}^n \]

by

\[ (x_0, x_1, x_2, x_3) \rightarrow x'_0 = x_0 x'. \]

Suppose \( P \) is a principal bundle over \( M \) with connection \( \nabla \) and curvature \( F \). Then \( p^* \nabla \) is an anti-self-dual connection on \( p^* P \) if

\[ *F = - [1/(2n - 1)] F \wedge \Omega^{n-1}, \]

i.e., \( F \) is \( c \)-self-dual.

**Proof:** In the \( x'_\mu \)-coordinates \( dx'_\mu \) is exactly the one-form \( \omega'_\mu \) of (2.4). Now a two-form \( F \) satisfies the equation

\[ *F = - [1/(2n - 1)] F \wedge \Omega^{n-1} \]

if and only if

\[ F = - \delta^* (F \wedge *F). \]

Using Theorem 2.2 we get the following equations

\[ F_{(\mu)} (\nu) = - F_{(\nu)} (\mu), \]

\[ F_{(\mu)} (\nu) = F_{(\mu)} (\nu) - F_{(\mu)} (\nu), \forall i, j, \]

\[ F_{(\mu)} (\nu) = F_{(\mu)} (\nu) \forall i, j, \alpha, \beta, \]

\[ F_{(\mu)} (\nu) = F_{(\mu)} (\nu) \forall i, j, \alpha, \beta, \alpha \neq \beta. \]

Let us denote the components of \( p^* F \) by \( F_{ab} \). As a consequence of the chain rule we get

\[ F_{ab} = \sum_{ij} x^i x^j F_{(i)} (\mu), \]

and therefore

\[ F_{01} = \sum_{ij} x^i x^j F_{(i)} (\mu) = - F_{23}, \]

\[ F_{02} = \sum_{ij} x^i x^j F_{(i)} (\mu) = F_{13}, \]

\[ F_{03} = \sum_{ij} x^i x^j F_{(i)} (\mu) = - F_{12}. \]

In other words, \( p^* \nabla \) is an anti-self-dual connection.

Recently, Pedersen and Poon used twistorial approach to find a generalization of the Bogomolny equations. They introduced Yang–Mills–Higgs equations \( R^4 \otimes R^n \). If one considers monopoles on \( R^4 \) as time invariant instantons on \( \mathbb{R}^4 \) the following simple geometric description of generalized monopoles comes with no surprise.

**Proposition 4.2:** Let \( x'_\mu, \mu = 0,1,2,3; i = 1,..., n \) be a global linear coordinate on \( \mathbb{R}^4 \otimes \mathbb{R}^n \) and let

\[ p: \mathbb{R}^4 \otimes \mathbb{R}^n \rightarrow \mathbb{R}^4 \otimes \mathbb{R}^n \]

be a projection

\[ (x'_0, x'_1, x'_2, x'_3) \rightarrow (x'_1, x'_2, x'_3). \]

If \( (\nabla, \Phi) \) is a generalized monopole then

\[ \Phi' = p^* \Phi + \sum_{i} \Phi' dx'_i. \]

(4.7)
is a $c_\ast$-self-dual connection. Conversely, any $c_\ast$-self-dual connection that is $x^i$-invariant determines a solution of the generalized monopole equation.

Proof: The curvature $F'$ of the connection $\nabla'$ is given by

$$F' = p^*F + \sum \nabla (\Phi') \wedge dx'_0 + \frac{1}{2} \sum [\Phi', \Phi'] \wedge dx'_0 \wedge dx'_0,$$

(4.8)

where $F$ is the curvature two-form of $\nabla$. Now, using Eqs. (4.4), we get

$$\nabla (\Phi') = F(\Phi') - F(\Phi'), \quad \forall i,j \quad (4.9)$$

$$\nabla (\Phi') = \nabla (\Phi'), \quad \forall i,j; \quad \alpha = 1,2,3,$$

which can be written as

$$\nabla (\Phi') = \nabla (\Phi'), \quad \forall i,j; \quad \alpha = 1,2,3.$$

(4.10)

The converse is obvious.

We can also obtain "monopole" analogs of $c$-self-duality equations in the $c_\ast$ and $c_\beta$ cases. The first one is not interesting, however, because it yields $n$ decoupled self-dual Bogomolny equations. In the second case we can explicitly write down the set of equations

$$\sum_{\alpha, \beta} \frac{1}{2} [\Phi', \Phi'] = \sum_{\alpha, \beta} F(\Phi') \wedge dx'_0 \wedge dx'_0,$$

(4.11)

For $n = 1$ these are just the usual Bogomolny equations with the reversed orientation. We do not know any nontrivial solutions of (4.11) for $n > 1$ at the moment. Finally, let us remark that we could introduce additional invariance and reduce the $c$-self-duality equation to $2n$ dimensions, assuming that the $c$-self-dual equations of $\mathbb{R}^4 \otimes \mathbb{R}^n$ be both $x^i$ and $x^j$ invariant. Then we obtain an analog of the well-known vortex equation of the two-dimensional Yang–Mills–Higgs theory. Again the $c_\ast$ case is the most natural generalization and we shall address this problem in a future work.

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