THE HITCHIN–THORPE INEQUALITY FOR EINSTEIN–WEYL MANIFOLDS

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ABSTRACT

An inequality relating the Euler characteristic, the signature and the L_2-norm of the curvature of the bundle of densities is proved for a four-dimensional compact Einstein–Weyl manifold. This generalises the Hitchin–Thorpe inequality for Einstein manifolds. The case where equality occurs is discussed and related to Hitchin's classification of Ricci-flat self-dual four-manifolds and to the recent work of Gauduchon on closed non-exact Einstein–Weyl geometries.

1. *Introduction*

The Hitchin–Thorpe inequality [**1**, **4**]

$$\chi(M) \geqslant \tfrac{3}{2}|\tau(M)|$$

is a relation between the signature τ and the Euler characteristic χ of a four-dimensional compact Einstein manifold. (Recall that a Riemannian manifold is called Einstein if the trace-free part of the Ricci curvature vanishes.) The idea behind the inequality is that in four dimensions the topological invariants τ and χ are given in terms of curvature, and therefore a Riemannian manifold with special curvature, such as an Einstein manifold, ought to satisfy some topological obstructions.

In this paper we shall prove a similar inequality for a class of *conformal* manifolds. A manifold M with conformal structure $[g]$ and torsion-free affine connection D compatible with the conformal structure is called a *Weyl manifold.* If we choose a metric g in the conformal structure, the compatibility is equivalent to the existence of a 1-form ω such that $Dg = \omega \otimes g$. The connection D is called the *Weyl connection.* The Weyl manifold $(M, [g], D)$ is said to be *Einstein–Weyl* if the trace-free part of the symmetric part of the Ricci curvature of the Weyl connection vanishes [**5**]. Thus, if $S(r_D)$ denotes the symmetric part of the Ricci curvature r_D of D, then the *Einstein–Weyl equations* in dimension four are

$$S(r_D) = \tfrac{1}{4}s_D\, g,$$

where s_D is the trace of r_D with respect to g and is called *the conformal scalar curvature.*

REMARKS. (1) If we consider a conformal change of metric $e^f g$, then the corresponding 1-form is $\omega + df$, the new volume is $e^{2f}\,\mathrm{vol}_g$ and the new norm on 2-forms is $e^{-f}|\cdot|_g$. It follows that the L_2-norm

$$\int_M |d\omega|_g^2\, \mathrm{vol}_g$$

is a conformal invariant. The new conformal scalar curvature is $e^{-f}s_D$, so the

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vanishing of the conformal scalar curvature is also a conformally invariant property. Indeed, in dimension four, the sign of the conformal scalar curvature is constant [**6**].

(2) If either the self-dual part W_+ or the anti-self-dual part W_- of the Weyl curvature vanishes, then the conformal manifold is said to be *conformally half-flat*.

(3) Note that $4d\omega$ is the curvature of the bundle of densities (see [**3**]).

THEOREM 1.1. *Let $(M,[g],D)$ be a compact oriented Einstein–Weyl manifold of dimension four, let g be a metric in the conformal structure, and let ω be the* 1*-form such that $Dg = \omega \otimes g$. Then the Euler characteristic $\chi(M)$ and the signature $\tau(M)$ satisfy the inequality*

$$\chi(M) \geqslant \tfrac{3}{2}|\tau(M)| + \frac{1}{32\pi^2}\int_M |d\omega|_g^2 \operatorname{vol}_g,$$

with equality if and only if the manifold M is conformally half-flat and has vanishing conformal scalar curvature.

After the proof of the theorem, we shall discuss the classification of geometries where we have equality and give new examples of manifolds which do not carry Einstein–Weyl structures.

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2. *Proof of Theorem* 1.1

On an oriented four-manifold (M,g), the curvature tensor decomposes into the following SO(4)-irreducible components [**1**]: the scalar curvature s_g, the trace-free part of the Ricci tensor $2B$, the self-dual Weyl tensor W_+ and the anti-self-dual Weyl tensor W_-. Using the Chern–Weil formula and the Hirzebruch signature formula, we can express the signature and the Euler characteristic by

$$\tau = \frac{1}{12\pi^2}\int_M (|W_+|_g^2 - |W_-|_g^2)\operatorname{vol}_g,$$

$$\chi = \frac{1}{8\pi^2}\int_M (\tfrac{1}{24}s_g^2 - 2|B|_g^2 + |W_+|_g^2 + |W_-|_g^2)\operatorname{vol}_g$$

(see [**1**, p. 371]).

In proving the theorem we shall make use of the conformal invariance by choosing the following metric which we call *the Gauduchon gauge.*

PROPOSITION 2.1 ([**2**]). *Up to a constant, there is a unique metric in the conformal class of a compact Weyl space such that the corresponding* 1*-form is co-closed.*

In the Gauduchon gauge, the vector field $\omega^\#$ dual to the 1-form is Killing [**8**], and it follows that the Ricci curvature r_g of the metric is given by

$$r_g = S(r_D) - \tfrac{1}{2}(\omega \otimes \omega - |\omega|_g^2 g)$$

(see [**5**]). Contracting this equation with ω and using the Einstein–Weyl equations, we obtain $r_g(\omega) = \frac{1}{4}s_D\omega$. Then the Bochner formula $r_g(\alpha) + \nabla^{*_g}\nabla\alpha = \Delta_g\alpha$ gives the following.

LEMMA 2.2. *In the Gauduchon gauge one has*

$$\Delta_g \omega = \tfrac{1}{2} s_D \omega,$$

where Δ_g *is the Laplacian of* g.

Thus, working in the Gauduchon gauge we obtain

$$s_g^2 = (s_D + \tfrac{3}{2}|\omega|_g^2)^2$$

and

$$4|B|_g^2 = |r_g - \tfrac{1}{2} s_g g|_g^2 = \left|\tfrac{1}{8}|\omega|_g^2 g - \tfrac{1}{2}\omega \otimes \omega\right|_g^2 = \tfrac{3}{16}|\omega|_g^4.$$

Substituting into the formulae for the Euler characteristic and the signature, and using Lemma 2.2, gives

$$\begin{aligned} 2\chi &= \frac{1}{4\pi^2}\int_M (\tfrac{1}{24}s_D^2 + \tfrac{1}{8}s_D|\omega|_g^2 + |W_+|_g^2 + |W_-|_g^2)\,\mathrm{vol}_g \\ &= \frac{1}{4\pi^2}\int_M \tfrac{1}{24}s_D^2 + \tfrac{1}{4}(\Delta_g\omega,\omega)_g + |W_+|_g^2 + |W_-|_g^2)\,\mathrm{vol}_g \\ &\geqslant 3\tau + \frac{1}{16\pi^2}\int_M |d\omega|_g^2\,\mathrm{vol}_g, \end{aligned}$$

with equality if and only if $s_D = 0$ and $W_- = 0$. If M is given the opposite orientation, it is still an Einstein–Weyl manifold, so we also have

$$2\chi \geqslant -3\tau + \frac{1}{16\pi^2}\int_M |d\omega|_g^2\,\mathrm{vol}_g,$$

with equality if and only if $s_D = 0$ and $W_+ = 0$. This completes the proof of Theorem 1.1.

3. *Remarks and examples*

If the 1-form is closed, then M is at least locally conformal to Einstein. This follows since ω is replaced by $\omega + df$ when the metric g is rescaled to $e^f g$. Thus, if the manifold is Einstein–Weyl but not locally conformal to Einstein, then $\chi > \tfrac{3}{2}|\tau|$.

From Lemma 2.2 we see that if the conformal scalar curvature vanishes, then ω is closed. If ω is also exact, $\omega = df$, we have $\Delta_g f = d^{*_g}\omega = 0$, so ω vanishes identically. Therefore, if the conformal scalar curvature vanishes, then either the Gauduchon gauge is Ricci-flat or $b_1(M) \geqslant 1$. (We have, in fact, $b_1(M) = 1$; see **[6]**.) Thus, we have equality in Theorem 1.1 if and only if either M is Ricci-flat and conformally half-flat (in which case M is flat, a $K3$ surface, an Enriques surface or the quotient of an Enriques surface by a free antiholomorphic involution **[4]**) or M is a conformally half-flat Einstein–Weyl manifold with closed non-exact 1-form. This type of Einstein–Weyl manifold has been studied intensively by Gauduchon **[3]**: for an Einstein–Weyl manifold with closed non-exact 1-form, the Gauduchon metric is isometric to the standard metric on $S^1 \times S^3$, $\omega^\#$ is equal to $\partial/\partial t$ and D is flat. Gauduchon calls these spaces *manifolds of type* $S^1 \times S^3$. Globally, they are of the form $(\mathbb{R} \times S^3)/\Gamma$, where Γ is the fundamental group. The map $\mathbb{R} \times S^3 \to \mathbb{R}^4\backslash\{0\}$, $(t,p) \mapsto e^t p$, is an isometry when $\mathbb{R} \times S^3$ is given the product metric and $\mathbb{R}^4\backslash\{0\}$ is given the metric g_0/r^2, where g_0 is the canonical flat metric on $\mathbb{R}^4$. The fundamental group is mapped into the conformal group of $\mathbb{R}^4$, the Weyl connection is mapped to the flat connection of $\mathbb{R}^4$, and $\omega^\#$ is given by $r(\partial/\partial r)$. If Γ is isomorphic to $\mathbb{Z}$, the manifold is said to be *primitive*, and such spaces are related to Hopf manifolds: they are isomorphic to the quotients of $(\mathbb{C}^2\backslash\{0\}, g_0/r^2, \omega^\# = r\,\partial/\partial r)$ by the transformations generated by $(z_1, z_2) \mapsto (\alpha z_1, \beta z_2)$, $\alpha, \beta \in \mathbb{C}$, $|\alpha| = |\beta| > 1$.

Let $M = n\mathbb{C}P(2)$ (the connected sum of n copies of $\mathbb{C}P(2)$). Then M is simply connected, and $\tau = n$, $\chi = n+2$; so applying Theorem 1.1 and the remark above, we obtain that if M is Einstein–Weyl then $n \leqslant 3$. Hence $n\mathbb{C}P(2)$ (for $n \geqslant 4$) is a simply connected compact manifold which does not carry an Einstein–Weyl structure (partial results in this direction were proved in [5, 6]).

Let M be a $K3$ surface. Then M can carry only Einstein–Weyl structures which are Einstein, and $M \# M$ carries no Einstein–Weyl structure. For other simply connected spin manifolds which do not admit Einstein–Weyl structures, see [5].

If the conformal scalar curvature is negative (recall that the sign is constant [6]), then any Einstein–Weyl manifold is necessarily Einstein [3, 7]. In [6], Einstein–Weyl structures which are not locally conformal to Einstein and have positive conformal scalar curvature were constructed on $S^2 \times S^2$ and on $\mathbb{C}P(2) \# \overline{\mathbb{C}P}(2)$.

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