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A RELATIVE DEFORMATION OF MOISHEZON TWISTOR SPACES

HENRIK PEDERSEN AND YAT SUN POON

1. Introduction

It is known that when the twistor space associated to a compact self-dual manifold X is Moishezon, then the self-dual conformal class contains a metric with positive constant scalar curvature [10] and the manifold X is homeomorphic to a connected sum of complex projective planes [2].

There are indeed examples of self-dual conformal classes on connected sums of complex projective planes having Moishezon twistor spaces. For example, the twistor space of the Fubini-Study metric on $\mathbf{CP}^2$ is a flag manifold [4]. There is a 1-parameter family of self-dual metrics on $2\mathbf{CP}^2 := \mathbf{CP}^2\#\mathbf{CP}^2$ whose twistor spaces are bimeromorphic to the intersection of two quadrics [9]. There are also Moishezon twistor spaces associated to self-dual conformal classes on connected sums of three copies of the complex projective plane [3, 11]. More generally, LeBrun found [5] a family of self-dual metrics on connected sums of the complex projective plane

$$n\mathbf{CP}^2 := \underbrace{\mathbf{CP}^2\#\ldots\#\mathbf{CP}^2}_{n \text{ times}}$$

such that the associated twistor spaces are Moishezon. This rich collection of examples is in sharp contrast to Hitchin's theorem stating that the only projective algebraic twistor spaces are the complex projective 3-space associated to the Euclidean 4-sphere and the flag manifold associated to the conformal class containing the Fubini-Study metric on $\mathbf{CP}^2$ [4]. For convenience, we call the self-dual conformal classes on $n\mathbf{CP}^2$ constructed by LeBrun in [5] the LeBrun structures and call the associated twistor spaces the LeBrun twistor spaces.

In any Moishezon space, there are plenty of effective divisors as the space is bimeromorphic to a projective algebraic space. On the other hand,

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a twistor space is foliated by a smooth family of rational curves, namely the real twistor lines [4]. The intersection number of a divisor in a twistor space with a real twistor line will be called the *degree* of the given divisor. Among all divisors in a Moishezon twistor space, we shall be concerned with the following two types:

Definition 1.1 [5]. A divisor is called an elementary divisor if it is a degree one divisor.

Definition 1.2 [11]. A divisor is called a fundamental divisor if it is linearly equivalent to $-\frac{1}{2}K$, where K is the canonical divisor.

Remark 1.3 [4]. Since the restriction of the canonical bundle onto a real twistor line is a degree -4 bundle, a fundamental divisor is a degree 2 divisor.

One of many features of the LeBrun twistor spaces is that they always contain effective elementary divisors [5, 11]. The algebraic structures of twistor spaces admitting such divisors are quite well understood [7, 11]. In general, although a Moishezon twistor space contains plenty of effective divisors, one does not know, a priori, the minimum possible degree of these effective divisors. As our first step to investigate Moishezon twistor spaces beyond the LeBrun spaces, we want to study Moishezon twistor spaces containing effective fundamental divisors. Our first concern is the availability of examples.

It is known that if the twistor space of a compact simply connected self-dual manifold contains an effective fundamental divisor, then X is diffeomorphic to $n\mathbf{CP}^2$ [8]. On $n\mathbf{CP}^2$ with $n \leq 3$, the twistor space of a self-dual conformal class is Moishezon if and only if the conformal class contains a metric with positive scalar curvature. Moreover, the algebraic structures of such twistor spaces can be described using the techniques in [4], [9], and [11]. Therefore, *we shall focus on the cases where* $n \geq 4$.

To look for twistor spaces on $n\mathbf{CP}^2$ containing effective fundamental divisors, we take a relative deformation of the LeBrun twistor spaces. It means that we are seeking a complex family of pairs (Z_t, S_t) of twistor spaces Z_t and effective fundamental divisors S_t contained in Z_t such that the central fibre of the deformation is a pair consisting of a LeBrun twistor space and a fundamental divisor in it. In our search for examples, our first result is

Theorem 2.3. *There exists a versal deformation of* (Z, S), *where* Z *is a LeBrun twistor space and* S *is a smooth effective fundamental divisor. Moreover, the complex dimension of the parameter space of this deformation family is* $5n-5$.

This result leads us to prove that there are interesting examples:

Theorem 3.6. *On $n\mathbf{CP}^2$, $n \geq 4$, there exists a small deformation of LeBrun twistor spaces containing an effective fundamental divisor but no effective elementary divisor.*

Theorem (3.6) fails to be true for $n \leq 3$. The reason is that a small deformation of a LeBrun conformal class again contains a metric with positive scalar curvature. Therefore, due to a combination of the Riemann-Roch formula and Hitchin's vanishing theorems [9], the twistor space always contains both effective elementary divisors and effective fundamental divisors. Indeed, it will be clear how our proof requires $n \geq 4$.

As we are searching for Moishezon twistor spaces, the next step we take is to calculate the algebraic dimension of the twistor spaces obtained by relative deformations. To our surprise, we find the following result:

Theorem 4.7. *Suppose that Z is a small relative deformation of a LeBrun twistor space on $n\mathbf{CP}^2$ and a nonsingular effective fundamental divisor. If Z is also a Moishezon space, then there is an effective elementary divisor on Z.*

This observation immediately raises a question

Question. If Z is Moishezon, does it necessarily contain an effective elementary divisor?

In closing we mention that Theorem 4.7 comes along with the following result:

Theorem 4.8. *Suppose that (Z, S) is a small relative deformation of a LeBrun twistor space on $n\mathbf{CP}^2$, $n \geq 4$, and a real irreducible fundamental divisor. If Z is Moishezon, then either Z is a LeBrun twistor space or Z is the twistor space of a conformally one-point compactification of an asymptotically Euclidean, scalar flat Kähler metric on a blow-up of $\mathbf{C}^2$ at 4 points such that three of them are collinear.*

2. Deforming LeBrun twistor spaces

In order to fix the notation we review the set-up in [6]. For a survey of general deformation theory which is related to twistor theory we also refer to [3]. Let Z denote a LeBrun twistor space on $n\mathbf{CP}^2$. The complete linear system of fundamental divisors $|-\frac{1}{2}K|$ in a LeBrun twistor space is three-dimensional. The image of the associated map of this linear system is a two-dimensional nonsingular quadric surface [5]. Therefore, there is a meromorphic projection $\pi : Z \longrightarrow \mathbf{CP}^1 \times \mathbf{CP}^1$ onto a quadric surface. This meromorphic morphism is not holomorphic as the system of fundamental divisors has a pair of rational curves C and $\bar{C}$ as its base locus. The curve $\bar{C}$ is the image of C under the real structure of Z. These two curves are disjoint from each other. After blowing-up this pair of curves, the meromorphic projection π is lifted to be a holomorphic projection $\hat{\pi}$: $\hat{Z} \longrightarrow \mathbf{CP}^1 \times \mathbf{CP}^1$, where $\hat{Z}$ is the blow-up of Z. Then $\hat{Z}$ is the resolution of a conical bundle $\tilde{Z}$ over $\mathbf{CP}^1 \times \mathbf{CP}^1$ with singularities, and Z is a small resolution of $\tilde{Z}$. Furthermore, any fundamental divisor S contains two real twistor lines and the pair of nonintersecting rational curves C and $\bar{C}$. From the adjunction formula on the twistor space we deduce that the canonical bundle K_S of any smooth irreducible fundamental divisor S is the restriction of $K^{1/2}$.

Lemma 2.1. *Let L denote a real twistor line contained in S. Then, as divisors on the surface S, we have*

$$-K_S = C + \bar{C} + 2L.$$

Furthermore, the cohomology groups $H^2(S, \mathbf{K}_S^{-1})$, $H^1(S, \mathcal{O}(L^2))$ and $H^2(S, \mathcal{O}(L^2))$ vanish. The Todd genus of S is equal to 1 and $c_1^2(S) = 8 - 2n$.

Proof. By Serre duality $h^2(S, \mathbf{K}_S^{-1}) = h^0(S, \mathbf{K}_S^2)$. The vanishing of the cohomology groups and the value of the Todd genus follow from the fact that S is a rational surface ruled by the system $|L|$ [1]. Moreover, from [4], we have

$$c_1^2(S) = \frac{1}{8} c_1^3(Z) = 8 - 2n.$$

Finally, since $-K_S$ is obtained by intersecting a divisor in $|-\frac{1}{2}K|$ with S, an element of $|-K_S|$ will consist of the two curves C and $\bar{C}$ plus the two curves above the conic intersections in $\mathbf{CP}^1 \times \mathbf{CP}^1$. q.e.d.

We collect the following facts from [6]. Let $\Theta_{C\bar{C}} := \Theta_C \cap \Theta_{\bar{C}}$ where $\Theta_C \subseteq \mathcal{O}(TZ)$ denotes the subsheaf of holomorphic vector fields which are tangent to C. Then

Proposition 2.2. *For $n \geq 3$, the dimension of the cohomology groups of the sheaf of holomorphic vector fields is given by*

$$h^j(Z, \mathcal{O}(TZ)) = \begin{cases} 1, & j = 0, \\ 7n - 14, & j = 1, \\ 0, & j \geq 2, \end{cases}$$

and $h^2(Z, \Theta_{C\bar{C}}) = 0$.

A *relative* version of the Kodaira-Spencer deformation theory for the pair (Z, S) is now available [6]. Let Θ_S be the subsheaf of holomorphic tangent vector fields which are tangent to a smooth hypersurface S. If the cohomology group $H^2(Z, \Theta_S)$ vanishes, there exists a deformation of the pair (Z, S) with tangent space isomorphic to $H^1(Z, \Theta_S)$ at the origin $Z_0 = Z$ of the deformation.

Theorem 2.3. *There exists a smooth versal deformation of* (Z, S), *where* Z *is any LeBrun twistor space and* S *is a smooth irreducible fundamental divisor. Moreover, the complex dimension of the parameter space of this family is* $5n - 5$.

Proof. Consider the short exact sequence

$$0 \longrightarrow \Theta_S \longrightarrow \mathcal{O}(TZ) \longrightarrow \mathcal{O}_S(\mathbf{K}^{-1/2}) \longrightarrow 0 \tag{2.4}$$

and the associated long exact sequence

$$\to H^1(Z, \mathcal{O}(TZ)) \to H^1(S, \mathbf{K}_{|S}^{-1/2}) \to H^2(Z, \Theta_S) \to H^2(Z, \mathcal{O}(TZ)).$$

From Proposition (2.2), we have $H^2(Z, \mathcal{O}(TZ)) = 0$. The vanishing of $H^2(Z, \Theta_S)$ follows if we can prove the surjectivity of the map

$$H^1(Z, \mathcal{O}(TZ)) \longrightarrow H^1(S, \mathbf{K}_{|S}^{-1/2}).$$

Note that this map is induced by the projection onto the normal bundle of S in TZ.

From Lemma (2.1), we have a short exact sequence of sheaves on S

$$0 \longrightarrow \mathcal{O}_S(L^2) \longrightarrow \mathcal{O}_S(\mathbf{K}_S^{-1}) \longrightarrow \mathcal{O}_{C \cup \bar{C}}(\mathbf{K}_S^{-1}) \longrightarrow 0.$$

Since the curve C and its conjugate in the twistor space are mutually disjoint, Lemma (2.1) implies that the induced long exact sequence of this short exact sequence yields the following isomorphism

$$H^1(S, \mathbf{K}_S^{-1}) \xrightarrow{\simeq} H^1(C, \mathbf{K}_{S|C}^{-1}) \oplus H^1(\bar{C}, \mathbf{K}_{S|\bar{C}}^{-1}).$$

Therefore, it suffices to prove that the projection

$$H^1(Z, \mathcal{O}(TZ)) \longrightarrow H^1(C, \mathbf{K}_{|C}^{-1/2}) \oplus H^1(\bar{C}, \mathbf{K}_{|\bar{C}}^{-1/2})$$

is surjective. Let $\mathbf{N}_C^S$ denote the normal bundle of C in S, etc. Then

$$0 \longrightarrow \mathbf{N}_C^S \longrightarrow \mathbf{N}_C^Z \longrightarrow \mathbf{K}_{|C}^{-1/2} \longrightarrow 0$$

is an exact sequence. As $H^2(C, \mathbf{N}_C^S)$ vanishes for the reason that C is one-dimensional, we get a surjective map

$$H^1(C, \mathbf{N}_C^Z) \longrightarrow H^1(C, \mathbf{K}_{|C}^{-1/2}),$$

and similarly for $\bar{C}$. Due to Proposition (2.2), $H^2(Z, \Theta_{C\bar{C}})$ vanishes. Then the exact sequence of sheaves on the twistor space Z

$$0 \longrightarrow \Theta_{C\bar{C}} \longrightarrow \mathscr{O}(TZ) \longrightarrow \mathscr{O}(\mathbf{N}_C^Z) \oplus \mathscr{O}(\mathbf{N}_{\bar{C}}^Z) \longrightarrow 0$$

gives a surjective map

$$H^1(Z, \mathscr{O}(TZ)) \longrightarrow H^1(C, \mathbf{N}_C^Z) \oplus H^1(\bar{C}, \mathbf{N}_{\bar{C}}^Z).$$

Thus the group $H^2(Z, \Theta_S)$ vanishes.

Next we are going to compute the complex dimension of the parameter space of the relative deformation. Consider the long exact sequence from (2.4)

$$0 \longrightarrow H^0(Z, \Theta_S) \longrightarrow H^0(Z, \mathscr{O}(TZ)) \longrightarrow H^0(S, \mathbf{K}_{|S}^{-1/2}).$$

On Z there is a natural $\mathbf{C}^*$-action in the fibres of the projection $Z \longrightarrow \mathbf{CP}^1 \times \mathbf{CP}^1$ and this action generates $H^0(Z, \mathscr{O}(TZ)) \cong \mathbf{C}$. Since S is obtained by pulling back a conic on $\mathbf{CP}^1 \times \mathbf{CP}^1$, the action preserves S so the map

$$H^0(Z, \Theta_S) \longrightarrow H^0(Z, \mathscr{O}(TZ))$$

is an isomorphism. Thus we have the exact sequence

$$0 \to H^0(S, \mathbf{K}_{|S}^{-1/2}) \to H^1(Z, \Theta_S) \to H^1(Z, \mathscr{O}(TZ)) \to H^1(S, \mathbf{K}_{|S}^{-1/2}) \to 0$$

which together with the Riemann-Roch formula on S and Lemma (2.1) gives

$$\begin{aligned} h^1(Z, \Theta_S) &= h^1(Z, \mathscr{O}(TZ)) + h^0(S, \mathbf{K}_S^{-1}) - h^1(S, \mathbf{K}_S^{-1}) \\ &= 7n - 14 + \chi(S, \mathbf{K}_S^{-1}) = 7n - 14 + c_1^2(S) + 1 = 5n - 5. \end{aligned}$$

As a contrast to Theorem (2.3), there is

Theorem 2.5 [6]. *There exists a smooth versal relative deformation of Z together with an elementary divisor, where Z is any LeBrun twistor space. Moreover, the complex dimension of the parameter space of this family is $5n - 8$.*

Thus one is led to believe that LeBrun twistor spaces can be deformed to contain effective fundamental divisors but no effective elementary divisors. However, as neither of the relative deformations are *universal,* a proof is needed and will be completed in the next section. After all, for $n \leq 3$, *any* small deformation of a Moishezon twistor space on $n\mathbf{CP}^2$ contains both effective elementary divisors and effective fundamental divisors [11]. As a preparation, we shall prove the following

Proposition 2.6. *Any deformation of an effective fundamental divisor S is induced by a relative deformation of (Z, S).*

Proof. To examine the deformation of the surface S, recall that the anticanonical system of a fundamental divisor in a LeBrun twistor space is nonempty. For instance, it contains the intersection of a fundamental divisor and a conjugate pair of elementary divisors. Therefore, we have the following exact sequence on the surface S:

$$0 \to \mathbf{K}_S \otimes \Omega_S \to \Omega_S \to \mathcal{O}_F(\Omega_S) \to 0$$

where Ω_S is the bundle of holomorphic 1-forms on S and $\mathcal{O}_F$ is the structure sheaf of an effective anticanonical divisor on S. Since S is a rational surface, there are no nontrivial global holomorphic 1-forms on it. The above exact sequence implies the vanishing of $H^0(S, \mathbf{K}_S \otimes \Omega_S)$. By Serre duality, $H^2(S, \mathcal{O}_S(TS)) = 0$. Therefore, the deformation of the surface S is versal, and the dimension of the parameter space of the versal deformation is given by the dimension of $H^1(S, \mathcal{O}_S(TS))$. Due to LeBrun's construction [5], the space of holomorphic vector field on the surface S is 1-dimensional. Applying the Riemann-Roch formula, we find that

$$H^j(S, \mathcal{O}_S(TS)) = \begin{cases} 1, & j = 0 \\ 4n-5, & j = 1, \\ 0, & j = 2. \end{cases} \tag{2.7}$$

As the deformation of S is versal, to prove our claim, it suffices to show that the natural map

$$R_2 : H^1(Z, \Theta_S) \to H^1(S, \mathcal{O}_S(TS))$$

induced by restriction is surjective.

Recall that N_C^S is the normal bundle of the curve C in the surface S. It is extended by zeros when it is considered as a coherent sheaf on the twistor space Z. The sheaf $N_{\overline{C}}^S$ is defined similarly. Let $\Theta_{C\overline{C}}^S$ be the sheaf of germs of holomorphic tangent vector fields on the twistor space which are tangent to S, C, and $\overline{C}$; i.e. $\Theta_{C\overline{C}}^S := \Theta_S \cap \Theta_C \cap \Theta_{\overline{C}}$. Let $\mathcal{J}$ be the ideal sheaf of the curve $C \cup \overline{C}$ in the surface S. Then we have the following commutative diagram of exact sequences of coherent sheaves on the twistor space Z:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{J}(\mathbf{K}^{-1/2}) & \longrightarrow & \mathcal{O}_S(\mathbf{K}^{-1/2}) & \longrightarrow & \mathcal{O}_{C\cup\overline{C}}(\mathbf{K}^{-1/2}) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Theta_{C\overline{C}} & \longrightarrow & \mathcal{O}(TZ) & \longrightarrow & N^Z_C \oplus N^Z_{\overline{C}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Theta^S_{C\overline{C}} & \longrightarrow & \Theta_S & \longrightarrow & N^S_C \oplus N^S_{\overline{C}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

From this diagram, we shall deduce that the induced map

$$R_1 : H^1(Z, \Theta_S) \to H^1(C, N^S_C) \oplus H^1(\overline{C}, N^S_{\overline{C}})$$

is surjective. Due to Proposition (2.2), $H^2(Z, \Theta_{C\overline{C}})$ vanishes. Note that the cohomology group $H^1(S, \mathcal{J}(\mathbf{K}^{-1/2}))$ also vanishes because as a divisor on the surface S, $-\frac{1}{2}K_{|S} = C + \overline{C} + 2L$, where L is the divisor class of a real twistor line contained in the surface S (Lemma 2.1). Therefore, $H^1(S, \mathcal{J}(\mathbf{K}^{-1/2})) = H^1(S, \mathcal{O}(L^2))$. The vanishing of the last cohomology group is given in Lemma (2.1). The vanishing of $H^2(Z, \Theta^S_{C\overline{C}})$ follows from the commutative diagram. Therefore, the restriction map R_1 is surjective.

The map $r_1 : \Theta_S \to N^S_C \oplus N^S_{\overline{C}}$ factors through $r_2 : \Theta_S \to \mathcal{O}_S(TS)$ and $r_3 : \mathcal{O}_S(TS) \to N^S_C \oplus N^S_{\overline{C}}$. Therefore, the map R_3 on the first cohomology group, induced by r_3, is also surjective. In order to prove that the induced map R_2 of r_2 is also surjective on the first cohomology group, it suffices to see that R_2 is surjective onto the kernel of R_3. Since C is a rational curve with self-intersection $-n$ on the surface S, the dimension of $H^1(C, N^S_C) \oplus H^1(\overline{C}, N^S_{\overline{C}})$ is equal to $2n-2$. Due to (2.7) and the fact that R_3 is surjective, the dimension of the kernel of R_3 is equal to $2n-3$. This is the dimension of the parameter space of the versal relative deformation of the pair $(S, C \cup \overline{C})$. This parameter space corresponds to the choices of n points on C and another n points on $\overline{C}$. As any of these choices of points is realized by an explicit deformation of LeBrun twistor spaces [5, 6], R_2 is surjective onto the kernel of R_3.

3. Structures of fundamental divisors

In this section, we shall investigate the structure of an irreducible fundamental divisor. When S is a real irreducible divisor of the system $|-\frac{1}{2}K|$, it is nonsingular [8]. Since $-\frac{1}{2}K \cdot L = 2$ for any real twistor line L, if S were not to contain any real twistor line, S would have been an unbranched double covering of the 4-manifold, $n\mathbf{CP}^2$ in the present case. As $n\mathbf{CP}^2$ is simply connected, S must contain a real twistor line L. By the adjunction formula,

$$\mathbf{K}_S^{-1} = \mathbf{K}_{|_S}^{-1/2}. \tag{3.1}$$

As $-\frac{1}{2}K \cdot L = 2$, L is a nonsingular rational curve of self-intersection 0 on the surface S. Then one may consider the associated map of the complete linear system $|L|$ on S. As $L^2 = 0$ on the surface S, this system is a pencil, and we have a morphism

$$S \longrightarrow \mathbf{P}(H^0(S, \mathcal{O}(L))^*) \cong \mathbf{CP}^1. \tag{3.2}$$

Therefore, S is a blow-up of a rational ruled surface. It is well known that a rational ruled surface is the total space of the projectivization of the bundle $\mathcal{O}(d) \oplus \mathcal{O}$ over $\mathbf{CP}^1$, for some $d \geq 0$ (known as a Hirzebruch surface). We shall denote such surface by Σ_d, and d is called the degree of the ruled surface. As $c_1^2(S) = \frac{1}{8}c_1^3(Z) = 8 - 2n$, when the twistor space Z is associated to $n\mathbf{CP}^2$, S is a blow-up of Σ_d $2n$-many times.

On the other hand, S and L are real with respect to the real structure on the twistor space Z; the vector space $H^0(S, \mathcal{O}(L))$ inherits a real structure, namely the complex conjugation. With respect to this real structure, the real points on $\mathbf{CP}^1$ are $\mathbf{RP}^1$, which is a circle. Note that this $\mathbf{CP}^1$ is the parameter space of the deformation of the real twistor line L in S. In particular, it is naturally a subspace of the full parameter space X^c of the deformation of L in the twistor space Z. Hence, the circle S^1 is naturally a submanifold of the 4-manifold X which parametrizes all real deformations of L on S. Therefore, the surface S contains an S^1-family of real twistor lines. With these structures on S in mind, we make a definition:

Definition 3.3. Suppose that S is the blow-up of a rational ruled surface. S is said to be real if there is an antiholomorphic involution without fixed point on S such that

(i) the anticanonical bundle is real,
(ii) the ruling has a real irreducible fiber, and
(iii) if a fiber of the ruling is real, it is irreducible.

The reasons we single out such properties of S to study are the following:

The meromorphic function field of a twistor space Z is generated by the sections of the fundamental line bundle and its positive powers [7, 10]. Therefore, if the algebraic dimension of Z is equal to 3, (3.1) implies that the anti-Kodaira dimension, $\kappa^{-1}(S)$, of the surface S is equal to 2. We shall use this fact when we impose conditions on the algebraic dimension in §4. Also, we shall now use the structures described in Definition (3.3) to study the configuration of blowing-up from Σ_d to S.

Let E_0 denote the divisor class of the zero section of the ruled surface Σ_d, L the class of a generic fiber, and E_i the exceptional divisor of the ith blow-up; then

$$-K_S = 2E_0 + (2-d)L - \sum_{i=1}^{2n} E_i ,$$

when S is the blow-up of Σ_d $2n$-many times. As E_{2n} is the last exceptional divisor, it is an irreducible rational curve with self-intersection -1. Let $\bar{E}_{2n}$ denote the conjugate curve; then it is also an irreducible rational curve. By the reality of L,

$$\bar{E}_{2n} \cdot L = E_{2n} \cdot L = 0.$$

Therefore,

$$\bar{E}_{2n} = kL - \sum_{i=1}^{2n} m_i E_i ,$$

for some integers k and m_i. As $-1 = \bar{E}_{2n}^2 = -\sum_{i=1}^{2n} m_i^2$, then $m_i = 0$, 1, or -1 for any i. On the other hand, both E_{2n} and $\bar{E}_{2n}$ are irreducible curves and they are conjugate to each other. As the real structure does not have any real point, $E_{2n} \cdot \bar{E}_{2n}$ is an even number. Therefore $m_{2n} = 0$. It means that E_{2n} and $\bar{E}_{2n}$ are a pair of mutually disjoint exceptional divisors of the first kind. Therefore, they can be simultaneously blown-down. Let S' be the blown-down surface. Since $\bar{E}_{2n}$ is disjoint from a generic fiber of the ruling as E_{2n} is, $\bar{E}_{2n}$ cannot be the infinity section. Therefore, S' is the blow-up of a ruled surface $2n-2$ times. Since E_{2n} is conjugate to $\bar{E}_{2n}$, S' inherits a real structure. For future reference, we shall consider S as an nth level surface and S' an $(n-1)$st level surface. To sum up our discussion in this section, we have

Lemma 3.4. *Any nth level real surface is obtained by blowing up a conjugate pair of points on an $(n-1)$st level real surface.*

When this blowing down process is carried out down to the 0th level, one obtains a real Hirzebruch surface. If the degree d of this minimal surface, Σ_d, is positive, then the infinity section is the only irreducible rational curve of negative self-intersection. Since $E_\infty \cdot L = 1$, the infinity section cannot be real. Therefore, the real minimal model of any real surface S is a quadric. In fact, the real minimal model of S is naturally identified as the space

$$\mathbf{Q} := L \times F$$

where $F \cong \mathbf{P}(H^0(S, \mathcal{O}(L))^*)$. The real structure on $L \times F$ is the antipodal map on L and the complex conjugate on F.

Lemma 3.5. *Any nth level real surface is obtained by blowing up a real quadric $2n$-many times at n pairs of conjugate points.*

As an application of our computations in the last two sections, we are now ready to prove the following:

Theorem 3.6. *On $n\mathbf{CP}^2$, $n \geq 4$, there exists a small deformation of LeBrun twistor spaces containing an effective fundamental divisor but no effective elementary divisor.*

Proof. Let (Z_1, S_1) be a pair of twistor space and effective fundamental divisor obtained by a small deformation of a pair LeBrun twistor space Z and its effective fundamental divisor S. If the twistor space Z_1 also contains a conjugate pair of effective elementary divisors D and $\overline{D}$, these divisors intersect S_1 along an anticanonical divisor. In particular, all the $2n$ points of blowing-up on S_1 are on the restriction of $D + \overline{D}$ on S_1. Since the $2n$ points are n pairs of conjugate points, there are n points on D and n points on $\overline{D}$. As D and $\overline{D}$ are conjugate to each other, using the real structure described as above, we see that there are n points on the divisor $L + F$ on the quadric surface $\mathbf{Q}$ and another n points on another divisor linearly equivalent to $L + F$. It means that this pair of n points are co-conic if we consider the divisor class $L + F$ as a conic. Clearly, when n is at least 4, we can always move n points on a quadric so that they are not co-conic. On the other hand, due to Proposition (2.6), such a surface can always be induced by a relative deformation of (Z, S). Thus, we can find a relative deformation (Z_1, S_1) of (Z, S) such that Z_1 does not contain any effective elementary divisor.

4. Algebraic dimensions

In a LeBrun twistor space over $n\mathbf{CP}^2$, any irreducible real surface S in the system $|-\frac{1}{2}K|$ is a blow-up of Σ_n. The curves C and $\bar{C}$ are the infinity section and its conjugate. To describe them on a blow-up of $L \times F$, we take a point p in L and its conjugate $\bar{p}$, then

$$F_p := \{p\} \times F; \qquad F_{\bar{p}} := \{\bar{p}\} \times F$$

is a conjugate pair of curves. Then a real irreducible fundamental divisor in a LeBrun twistor space is obtained by blowing-up n-many distinct points, $p_1, \cdots, p_n$ on F_p and their conjugates, $\bar{p}_1, \cdots, \bar{p}_n$ on $F_{\bar{p}}$. As the quadric surface $\mathbf{Q}$ can be embedded in $\mathbf{CP}^3$ such that the rational curves L and F are lines in $\mathbf{CP}^3$, we can consider the points $\{p_1, \cdots, p_n\}$ on the quadric $\mathbf{Q} = L \times F$ as collinear.

As any small deformation of a real S is a blow-up of a real quadric, any small deformation of (Z, S) induces a small deformation of the collinear configuration described in the previous paragraph. Since $p_1, \cdots, p_n$ are distinct, i.e., not infinitely near each other, after a small deformation, they remain distinct. However, they may not be collinear. The following lemma is the basic technical tool in our calculation:

Lemma 4.1. *Let q_1, q_2, q_3, q_4 be any four points in the collection of distinct points $\{p_1, p_2, p_3, \cdots, p_n\}$. If $\mathbf{Q}$ is blown-up at these four points and at their conjugate points $\bar{q}_1, \bar{q}_2, \bar{q}_3, \bar{q}_4$, then the resultant real surface S has $\kappa^{-1}(S) = 2$ if and only if at least three of the four points $\{q_1, q_2, q_3, q_4\}$ are collinear.*

Proof. (i) When a quadric is blown-up 8 times to a surface S, the anticanonical system $|-K_S|$ on the surface S is nonempty. When we choose a real blow-up, $|-K_S|$ contains a real element. When E is an irreducible real element contained in $|-K_S|$, it is a nonsingular elliptic curve [3, Proposition (4.3)(iv)]. Moreover, $\mathbf{K}_S^{-1}$ is trivial on E. Therefore, we have the exact sequence

$$0 \to \mathbf{K}_S^{-(m-1)} \to \mathbf{K}_S^{-m} \to \mathcal{O}_E \to 0\,.$$

From the long exact sequence of cohomology, we deduce inductively that

$$h^0(S, \mathbf{K}_S^{-m}) \leq m + 1.$$

Therefore, $\kappa^{-1}(S) \leq 1$. With our assumption that $\kappa^{-1}(S) = 2$, any real element in $|-K_S|$ is reducible. In particular, any real element in $|-K_S|$ can be blown down to be a real reducible element in $|2L + 2F|$ in the quadric $\mathbf{Q}$.

(ii) If $A+B$ is a real reducible element in $|2L+2F|$ such that A is real, then the intersection numbers $A \cdot L$ and $A \cdot F$ are both even. It follows that either $A = 2L$, $B = 2F$, or $A = 2F$, $B = 2L$. Therefore, the eight points of blowing-up are on the union of two lines and two fibers. Since the four points $\{q_1, q_2, q_3, q_4\}$ are obtained by a small deformation of four collinear distinct points, no two of them are co-fiber. Therefore, two fibers, i.e., $2L$ can contain at most $\{q_1, q_2\} \cup \{\bar{q}_1, \bar{q}_2\}$. Then the two lines, i.e., the $2F$ part, must contain $\{q_3, q_4\} \cup \{\bar{q}_3, \bar{q}_4\}$. In a small deformation, none of $\bar{q}_i$ and q_j can be collinear for any i and j. Therefore q_3, q_4 are collinear.

Suppose that the line containing q_3, q_4 does not contain any of q_1 and q_2, then the proper transform of these two lines and two fibers is a collection of four irreducible rational curves C_1, C_2, C_3, C_4 such $C_1 + C_2 + C_3 + C_4 = -K_S$ and $-K_S \cdot C_i = 0$. Therefore, we have the exact sequence

$$\mathcal{O} \to \mathbf{K}_S^{-(m-1)} \to \mathbf{K}_S^{-m} \to \mathcal{O}_{\sum_{i=1}^4 C_i} \to \mathcal{O}. \tag{4.2}$$

Since the ûnion of these four curves is a connected set,

$$h^0(S, \mathcal{O}_{\sum_{i=1}^4 C_i}) = 1 .$$

Then the induced long exact sequence of (4.2) implies that

$$h^0(S, \mathbf{K}_S^{-m}) \leq m + 1 ,$$

i.e., $\kappa^{-1}(S) \leq 1$. Therefore, the line containing q_3, q_4 must contain at least one of the q_1 and q_2. It means that at least three of the four points $\{q_1, q_2, q_3, q_4\}$ are collinear.

(iii) If $A + B$ is a real reducible element in $|2L + 2F|$ such that A is conjugate to B, i.e., $\bar{B} = A$, then

$$A = L + F , \qquad \bar{A} = B = L + F .$$

If A is reducible to be a sum $C + D$, then one may regroup $A + \bar{A}$ into $(C + \bar{C}) + (D + \bar{D})$. This is the case already treated in the last paragraph. Therefore, $A, \bar{A}$ can be assumed to be irreducible. Then they are irreducible plane conics in the quadric $\mathbf{Q}$. Their proper transforms form a pair of irreducible curves, E and $\bar{E}$, such that $E + \bar{E} = -K_S$. Moreover, $E \cdot \bar{E} = 0$ or $E \cdot \bar{E} = 2$.

When $E \cdot \bar{E} = 0$, then $E^2 = \bar{E}^2 = 0$. We have the exact sequences:

$$0 \to \mathbf{E}^{m-1}\bar{\mathbf{E}}^m \to \mathbf{K}_S^{-m} \to \mathcal{O}_E \to 0;$$
$$0 \to \mathbf{K}_S^{-(m-1)} \to \mathbf{E}^{m-1}\bar{\mathbf{E}}^m \to \mathcal{O}_E \to 0.$$

It follows that

$$h^0(S, \mathbf{K}_S^{-m}) \le h^0(S, \mathbf{K}_S^{-(m-1)}) + 2 \le 2(m+1).$$

It follows that $\kappa^{-1}(S) \le 1$.

Similarly, if $E\bar{E} = 2$, then $E^2 = \bar{E}^2 = -2$. We have

$$0 \to \mathbf{E}^{m-1}\bar{\mathbf{E}}^m \to \mathbf{K}_S^{-m} \to \mathscr{O}_E \to 0;$$
$$0 \to \mathbf{K}_S^{-(m-1)} \to \mathbf{E}^{m-1}\bar{\mathbf{E}}^m \to \mathscr{O}_E(-2) \to 0.$$

Then

$$h^0(S, \mathbf{K}_S^{-m}) \le h^0(S, \mathbf{K}_S^{-(m-1)}) + 1 \le m+1.$$

It follows that $\kappa^{-1}(S) \le 1$.

Therefore, when $\kappa^{-1} = 2$, any real element of the anticanonical system of an effective fundamental divisor is described in paragraph (ii). And at least three of the four points $\{q_1, q_2, q_3, q_4\}$ are collinear.

Corollary 4.3. *For $n \ge 4$, at least $(n-1)$ many of $\{p_1, p_2, \cdots, p_n\}$ are collinear if $\kappa^{-1}(S) = 2$.*

Proof. Since blowing-down map will not reduce κ^{-1}, taking any four points of $\{p_1, p_2, \cdots, p_n\}$, we can apply the last lemma.

Suppose that the collection of points are not collinear. Let p_k be the first point such that it is not on the line of $\{p_1, \cdots, p_{k-1}\}$. After possibly a relabelling, by the last lemma, $k \ge 4$. For any $j \ge k+1$. Consider $\{p_1, p_2, p_k, p_j\}$. Remember that the lines in question are lines of one family on a quadric surface. In particular, they are mutually disjoint. If there is a line passing through three of $\{p_1, p_2, p_k, p_j\}$, it must intersect the line containing p_1 and p_2 and hence is the line containing p_1 and p_2. As p_k is not on the line of p_1 and p_2, p_j must be on the line of p_1 and p_2. Therefore, all points except p_k are collinear.

After proving Corollary (4.3), we have two cases to study when we assume that the twistor space is Moishezon and that it is a small deformation of a LeBrun twistor space containing an effective fundamental divisor.

(i) All points are collinear.

In this case, the anticanonical system of S has two fixed components, namely the proper transform of the conjugate pair of lines on the quadric $\mathbf{Q}$ containing the conjugate pair of points $\{p_1, \cdots, p_n\}$ and $\{\bar{p}_1, \cdots, \bar{p}_n\}$. Then the moveable part of the system is $|2L|$. In particular, there is an S^1-family of real elements in $|-K_S|$ containing a real fiber L to order 2. By the exact sequence

$$0 \to \mathscr{O} \to \mathscr{O}(\mathbf{K}^{-1/2}) \to \mathscr{O}_S(\mathbf{K}_S^{-1}) \to 0, \tag{4.4}$$

and the vanishing of $H^1(Z, \mathscr{O})$, this family of real elements is given by a family of real elements in $|-\frac{1}{2}K|$ containing a real twistor line as singularity. As such an element in $|-\frac{1}{2}K|$ must be reducible [8, Lemma 2.1], we have at least a pencil of effective elementary divisors. By [11, Theorem 3.1], it can occur only if the twistor space Z is a LeBrun twistor space.

(ii) All but one point are collinear.

In this case, $|-K_S|$ has precisely four components:

C: the proper transform of the line through all of $\{p_1, \cdots, p_n\}$ but, say, p_n.

$\bar{C}$: the proper transform of the line through all of $\{\bar{p}_1, \cdots, \bar{p}_{n-1}\}$ but $\bar{p}_n$.

A: the proper transform of the fiber through p_n.

$\bar{A}$: the proper transform of the fiber through $\bar{p}_n$.

The intersection matrix of $\{C, \bar{C}, A, \bar{A}\}$ on S is

$$\begin{pmatrix} -(n-1) & 0 & 1 & 1 \\ 0 & -(n-1) & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}. \tag{4.5}$$

Suppose that s is a real holomorphic section of $\mathbf{K}_S^{-m}$. Let k and ℓ be the order of zeros on C and A respectively. By reality, the order of zeros on $\bar{C}$ and $\bar{A}$ is k and ℓ. Then

$$\begin{aligned} J &= -mK_S - k(C+\bar{C}) - \ell(A+\bar{A}) \\ &= (m-k)(C+\bar{C}) + (m-\ell)(A+\bar{A}) \end{aligned}$$

is effective, if not linearly equivalent to zero. Moreover, by the definition of k and ℓ,

$$J \cdot C \geq 0 \quad \text{and} \quad J \cdot A \geq 0.$$

From the intersection matrix (4.5), we have

$$\begin{aligned} &2(m-\ell) \geq (n-1)(m-k); \\ &2(m-k) \geq (m-\ell). \end{aligned} \tag{4.6}$$

For $n \geq 6$, it is possible only when

$$m = k = \ell,$$

i.e.

$$h^0(S, \mathbf{K}_S^{-m}) = 1,$$

for all $m \geq 0$. Then $\kappa^{-1}(S) = 0$.

For $n = 5$, it is possible only when the inequalities in (4.5) are equalities; i.e., $J \cdot C = 0$ and $J \cdot A = 0$. Therefore, the order of zeros of s along C and A are constants. As C and A intersect, then $k = \ell$. By (4.5), $m = k = \ell$. Therefore, we have $\kappa^{-1}(S) = 0$ again.

For $n = 4$, let p be the intersection of C and A, q the intersection of C and $\bar{A}$. Let L_p be the real twistor line through p and $\bar{p}$. L_q is similarly defined. Since $C+C+A+\bar{A} \in |-K_S|$, and $h^0(S, K_S^{-1}) = 1$. The exact sequence (4.4) shows that $|-\frac{1}{2}K|$ is a pencil. Also, $C + \bar{C} + A + \bar{A}$ is precisely the base locus of this pencil. Let $|-\frac{1}{2}K|_{L_p}$ be the system of fundamental divisors containing L_p. This is non-empty because there is an element S' in $|-\frac{1}{2}K|$ passing through a point z on L_p. Yet S' must also pass through p and $\bar{p}$ because they are in the base locus. Therefore, S' contains L_p. As L_p is real, so is the system $|-\frac{1}{2}K|_{L_p}$. Therefore, we can choose S' to be real. As S' contains L_p, C, and A, and the linear span of $T_p(L_p)$, T_pC, and T_pA is 3-dimensional, S' vanishes at p to order 2; i.e., S' is singular at p and $\bar{p}$ and hence is reducible [8, Lemma 2.1], to be a sum of a conjugate pair of effective elementary divisors $D_p + \bar{D}_p$. Similarly, we obtain $D_q + \bar{D}_q$. Without loss of generality, we may assume that D_p contains the curve C. Then this is an elementary divisor obtained by blowing up $\mathbf{CP}^2$ four times such that three of the four points are collinear. Due to [11, Theorem 7.11], when a twistor space contains such an elementary divisor, it is Moishezon.

In conclusion, we have proved the following result when $n \geq 4$:

Theorem 4.7. *Suppose that Z is a small relative deformation of a LeBrun twistor space on $n\mathbf{CP}^2$ and a nonsingular effective fundamental divisor. If Z is also a Moishezon space, then there is an effective elementary divisor on Z.*

When $n \leq 3$, the statement of Theorem (4.7) is also correct. The reason is that a self-dual conformal class must contain a metric with positive constant scalar curvature when the twistor space is Moishezon [10]. Therefore, the self-dual conformal class corresponding to a small deformation of a Moishezon twistor space also contains a metric with positive constant scalar curvature. Using Hitchin's vanishing theorems [4, 9] together with the Riemann-Roch formula, one can easily show that the twistor space associated to such a self-dual conformal class on $n\mathbf{CP}^2$, $n \leq 3$, always contains effective elementary divisors as well as effective fundamental divisor. In fact, the twistor space of such a self-dual conformal class is always Moishezon [10].

Furthermore, in the case described in paragraph (i), the twistor space Z is a LeBrun twistor space. In the case described in paragraph (ii), the twistor space is associated to a self-dual conformal structure on $4\mathbf{CP}^2$ such that it contains an effective elementary divisor obtained by blowing-up $\mathbf{C}^2$ four times with three points collinear. Therefore, we also have the following refinement of Theorem (4.7):

Theorem 4.8. *Suppose that (Z, S) is a small relative deformation of a LeBrun twistor space and a real irreducible fundamental divisor. If Z is Moishezon, then either Z is a LeBrun twistor space or Z is the twistor space of a conformally one point compactification of an asymtotically Euclidean, scalar flat Kähler metric on a blow-up of $\mathbf{C}^2$ at 4 points such that three of them are collinear.*

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Department of Mathematics and Computer Science, Odense University, Campusvej 55, Odense M, DK-5230, Denmark

Department of Mathematics, University of California at Riverside, Riverside, California 92521
E-mail address: ypoon@math.ucr.edu