Math. Ann. 282, 621–627 (1988)

Mathematische Annalen

Algebraic Dimension of Twistor Spaces

Y. Sun Poon*

Department of Mathematics, SUNY at Stony Brook, Stony Brook, NY 11794, USA

In [8], Hitchin proved that a compact Kählerian twistor space is in fact a projective algebraic manifold. Moreover, it is the twistor space associated to either the Euclidean 4-sphere S^4 or the complex projective plane $\mathbb{CP}^2$ with Fubini-Study metric. The twistor spaces are $\mathbb{CP}^3$ and the flag of lines in $\mathbb{CP}^2$ respectively. In [10, 11], the author proved the existence of self-dual metric with positive scalar curvature on the connected-sums of two or three copies of the complex projective planes. Their twistor spaces are Moishézon spaces. In fact, they are the small resolution of the intersection of two quadrics in $\mathbb{CP}^5$ with four nodes and the double covering of $\mathbb{CP}^3$ branched along a quartic with thirteen nodes. Recently, the joint work of Donaldson and Friedman [3] produced a general procedure to construct new twistor spaces and hence self-dual manifolds. In this article, we shall follow the spirit of Hitchin's work and prove the following:

Theorem. *If the twistor space of a compact self-dual manifold is Moishézon, the self-dual conformal class contains a metric with positive scalar curvature.*

After a brief introduction, we shall apply a theorem of Grauert to show that the sections of a sufficiently large power of the anticanonical bundle on the twistor space generate the meromorphic function field when the twistor space is Moishézon. This piece of information will enable us to apply a Bochner type argument to prove the existence of metric with positive scalar curvature in the given conformal class.

For a given self-dual manifold X, the twistor space Z is the total space of the sphere bundle of the anti-self-dual two forms. It has a naturally defined complex structure determined by the self-dual conformal class [1]. The fibres of the fibration from Z onto X are nonsingular rational curves. The antipodal map on each fibre is the restriction of an antiholomorphic involution τ on Z. Of course, it has no fixed points. It is the so-called real structure on Z. Objects that are invariant under the real structure are said to be real. For example, the fibres from Z onto X

* The author is partially supported by a grant of the National Science Foundation
Current address: Department of Mathematics, Rice University, Houston, TX 77251, USA

are called real twistor lines, the canonical bundle K of Z is real. There is always a holomorphic line bundle such that its second power is K. This bundle is denoted by $K^{1/2}$. The obstruction to the bundle $K^{1/2}$ to have nontrivial index is the second Stiefel Whitney class of X. Since the twistor space is determined only by the conformal class of a metric [7], twistor correspondence can associate certain holomorphic objects on the twistor space to conformal invariant objects on the manifold X. For example, when S_+^m and S_-^m are the m-th symmetric power of the $+1/2$ and $-1/2$ spinor bundles, there are conformal invariant differential operators given by the composition of orthogonal projection and covariant derivatives [1, 7]:

$$D_m: S_-^m \to S_-^m \otimes S_- \otimes S_+ \to S_-^{m-1} \otimes S_+$$

$$\bar{D}_m: S_-^m \to S_-^m \otimes S_- \otimes S_+ \to S_-^{m+1} \otimes S_+ .$$

Here we are using the fact that the (complexified) tangent bundle of X is isomorphic to $S_- \otimes S_+$ and that

$$S_-^m \otimes S_- \otimes S_+ = (S_-^{m-1} \otimes S_+) \otimes (S_-^{m+1} \otimes S_+)$$

is an orthogonal irreducible decomposition. Then the twistor correspondence gives the following natural isomorphisms:

$$\ker D_m = H^1(Z, \mathcal{O}(K^{\frac{m}{4}+\frac{1}{2}})) \tag{1}$$

and

$$\ker \bar{D}_m = H^0(Z, \mathcal{O}(K^{-\frac{m}{4}})) . \tag{2}$$

D_m is the Dirac operator; $\bar{D}_m$ is the twistor operator. Of course, if X is not a spin manifold, m has to be an even integer.

Suppose that Z is a Moishézon space, in other words, the algebraic dimension of Z is equal to the its dimension. Since Z is three dimensional, it can be blown-up, with a finite collection of points and/or nonsingular curves on Z as centres, to a projective algebraic 3-fold W [9]. Let $\bar{W}$ be the blowing-up of Z with conjugate centres. Since exceptional divisors of blowing-up are simply the projectivization of the normal bundles, the differential of the real structure τ induces an anti-holomorphic map:

$$\tau : \bar{W} \to W ,$$

such that $\tau \circ p = \bar{p}$, where p and $\bar{p}$ are the blowing-down maps from W and $\bar{W}$ onto Z respectively. Let ϱ be the inverse of τ, then ϱ is an antiholomorphic map such that

$$\varrho \circ \bar{p} = p , \qquad \tau \circ \varrho = \text{identity} , \qquad \varrho \circ \tau = \text{identity} .$$

Since W is projective algebraic, there is a positive line bundle L on W. Let $\{U_\alpha : \alpha \in A\}$ be a good covering on W such that the transition function of the bundle L with respect to this covering is given by $\{g_{\alpha\beta}\}$. Then $\{\varrho(U_\alpha) : \alpha \in A\}$ forms a covering of $\bar{W}$. Let

$$h_{\alpha\beta}(z) := \overline{g_{\alpha\beta} \circ \tau(z)} .$$

Then $\{h_{\alpha\beta}\}$ is the transition functions of a holomorphic line bundle $\bar{L}$ on $\bar{W}$ with respect to the covering $\{\varrho(U\hat{\alpha})\}$. From this description, it is obvious that if Θ is a positive (1, 1)-form representing the first Chern class of L on W, then $\overline{\tau^*\Theta}$ is a positive (1,1)-form representing the first Chern class of bundle $\bar{L}$. Therefore, $\bar{W}$ is also projective algebraic. Let V be the fibre product of W and $\bar{W}$ over Z. As a closed complex subvariety of the product of the two projective algebraic manifolds W and $\bar{W}$, V is a projective algebraic variety. However, on V, there is an induced real structure inherited from $W\times\bar{W}$ defined by:

$$\tilde{\tau}:(v,\bar{w})\mapsto(\varrho(\bar{w}),\tau(v)).$$

This antiholomorphic involution on $W\times\bar{W}$ leaves V invariant. In fact, when V is considered as a variety that is blown down to the given twistor space, the real structure $\tilde{\tau}$ on V is a lifting of the real structure on Z. Obviously, we do not know, in general, if V is nonsingular or not. Neither do we know if the real structure on V has fixed point or not.

Since V is projective algebraic, there is a very ample line bundle F on V. Then $\tilde{\tau}^*\bar{F}$ is also a very ample line bundle because $\tilde{\tau}$ is an antiholomorphic involution. The space of global sections of F is conjugate linearly isomorphic to the space of global sections of $\tilde{\tau}^*F$. As the tensor product of two very ample line bundles is again very ample, there are *real* very ample line bundles on V, $F\otimes\tilde{\tau}^*\bar{F}$ for instance.

On the other hand, when $\mathscr{F}$ is any sheaf on V, there are the direct image sheaves $R^i\pi_*\mathscr{F}$ on the twistor space Z. The 0-th direct image sheave will be denoted by $\pi_*\mathscr{F}$. Since Z and V are compact complex manifolds, a theorem of Grauert [5, 6] on proper morphisms of complex analytic spaces states that:

When $\mathscr{F}$ is a very ample invertible sheave, then

(i) for all i, n, $R^i\pi_*\mathscr{F}^n$ is a coherent sheaf

(ii) for $i>0$ and $n\gg 0$, $R^i\pi_*\mathscr{F}^n=0$.

Let us choose a very ample real line bundle F_0 on V. Denote $\mathcal{O}_V(F_0)$ by $\mathscr{F}_0$. Let $F=F_0^n$ be a positive power of F_0 with n so big that the second assertion in the above theorem is true for $\mathscr{F}=\mathcal{O}_V(F)=\mathscr{F}_0^n$, i.e.

$$R^i\pi_*\mathscr{F}=0 \quad \text{for all} \quad i>0.$$

Since all the higher direct image sheaves of $\mathscr{F}$ vanish, a spectral sequence argument [4, 6] shows that there is a natural isomorphism

$$H^i(V,\mathscr{F})\cong H^i(Z,\pi_*\mathscr{F}) \tag{3}$$

for all $i\geqq 0$. As when we replace $\mathscr{F}$ by its positive power $\mathscr{F}^m$, $R^i\pi_*\mathscr{F}^m$ also vanishes by Grauert's theorem, we actually have the natural isomorphisms:

$$H^i(V,\mathscr{F}^m)\cong H^i(Z,\pi_*\mathscr{F}^m) \tag{4}$$

for all $i\geqq 0$, $m\geqq 1$.

Note that each $\pi_*\mathscr{F}^m$ is coherent. Since $\mathscr{F}^m$ is actually locally free, $\pi_*\mathscr{F}^m$ is torsion free. As a consequence, the set A_m on which $\pi_*\mathscr{F}^m$ fails to be free is an analytic subspace of Z of codimension at least 2. In other words, the dimension of A_m is at most 1.

Let $\mathscr{D}$ be the bidual sheaf of $\pi_*\mathscr{F}$, then $\mathscr{D}$ is locally free. As $\pi_*\mathscr{F}$ has rank 1, $\mathscr{D}$ is the sheaf of germs of sections of a holomorphic line bundle D on the twistor space

Z. Then the bidual sheaf of $\pi_*\mathscr{F}^m$ is $\mathcal{O}(D^m)=\mathscr{D}^m$. Since $\pi_*\mathscr{F}^m$ is torsion free, there is a natural injection:

$$0\to\pi_*\mathscr{F}^m\to\mathscr{D}^m.$$

Let $\mathscr{Q}_m$ be the quotient sheaf of $\pi_*\mathscr{F}^m$ in $\mathscr{D}^m$. We have an exact sequence of coherent sheaves on Z:

$$0\to\pi_*\mathscr{F}^m\to\mathscr{D}^m\to\mathscr{Q}_m\to 0 \tag{5}$$

Since $\pi_*\mathscr{F}^m$ is free except on the set A_m, $\mathscr{Q}_m$ is supported by A_m. As the dimension of A_m is at most 1, Grothendieck's vanishing theorem [6] implies that

$$H^j(Z,\mathscr{Q}_m)=0 \quad \text{for} \quad j=2,3.$$

Combining the natural isomorphism (4) with the induced long exact sequence of (5), we have

$$\begin{aligned}0&\to H^0(V,\mathscr{F}^m)\to H^0(Z,\mathscr{D}^m)\to H^0(Z,\mathscr{Q}_m)\\ &\to H^1(V,\mathscr{F}^m)\to H^1(Z,\mathscr{D}^m)\to H^1(Z,\mathscr{Q}_m)\\ &\to H^2(V,\mathscr{F}^m)\to H^2(Z,\mathscr{D}^m)\to 0\\ &\quad H^3(V,\mathscr{F}^m)\xrightarrow{\cong} H^3(Z,\mathscr{D}^m).\end{aligned}$$

Moreover, $\mathscr{F}$ is a positive power of the ample sheaf $\mathscr{F}_0$ on V. When this power is sufficiently large, a generalization of Kodaira's vanishing theorem [6] shows that, for all $m\geqq 1, j\geqq 1$,

$$h^j(V,\mathscr{F}^m)\equiv\dim H^j(V,\mathscr{F}^m)=0. \tag{6}$$

Therefore, we have

$$0\to H^0(V,\mathscr{F}^m)\to H^0(Z,\mathscr{D}^m)\to H^0(Z,\mathscr{Q}_m)\to 0 \tag{7}$$

$$h^2(Z,\mathscr{D}^m)=h^3(Z,\mathscr{D}^m)=0. \tag{8}$$

From the Riemann-Roch formula and the ampleness of $\mathscr{F}$, we can see that $\chi(V,\mathscr{F}^m)$ is a cubic polynomial $p(m)$ in m with positive coefficient at the leading term m^3. Then the equations (6) show that

$$h^0(V,\mathscr{F}^m)=p(m).$$

The exactness of (7) shows that

$$h^0(Z,\mathscr{D}^m)=g(m), \tag{9}$$

where $g(m)$ is a cubic polynomial with positive coefficient at m^3.

So far, we have been working on general compact Moishézon space. However, as we choose F to be real bundle on V, D is a positive power of the fundamental line bundle $K^{-1/2}$. The reason is the following:

A compact Moishézon space has Hodge symmetry [13]. But a twistor space has no nontrivial holomorphic forms [8]. Therefore, the Hodge numbers have to satisfy the following:

$$h^{0,2}=h^{2,0}=0, \qquad h^{0,1}=h^{1,0}=0.$$

Then the induced cohomology exact sequence of the exponential sequence shows that the Picard group is isomorphic to the additive group $H^2(Z,\mathbb{Z})$ via the first Chern class.

Now on V, F is a real line bundle in the sense that $\tilde{\tau}^*\bar{F}=F$. However, $\tilde{\tau}$ is a lifting of the real structure on Z. Therefore, $\pi\cdot\tilde{\tau}=\tau\cdot\pi$, i.e. $\tau^*\pi_*\bar{\mathscr{F}}=\pi_*\mathscr{F}$ and hence $\tau^*\bar{D}^m=D^m$. In other words, D is a real line bundle on Z and $\tau^*c_1(D)=c_1(D)$. By the Leray-Hirsch theorem, one can check that $H^2(Z,\mathbb{R})$ is generated by c_1, the first Chern class of the twistor space, and the $H^2(X,\mathbb{R})$ via the projection p from the twistor space onto X. Therefore

$$c_1(D)=kc_1+p^*\alpha\,,$$

where k is a number and α is in $H^2(X,\mathbb{R})$. However, $\tau^*c_1=c_1$ and $\tau^*\alpha=-\alpha$, the reality of $c_1(D)$ implies that

$$c_1(D)=kc_1\,.$$

As we have the isomorphism between the Picard group and $H^2(Z,\mathbb{Z})$, the above equality implies that

$$D=K^{-k}\,.$$

As the restriction of K^{-1} on every real twistor line is the fourth power of the hyperplane bundle [8], Eq. (9) shows that k is positive. Note that k can be a half-integer. The consequences of Eqs. (8) and (9) are

$$h^0(Z,K^{-mk})=g(m)=0(m^3) \tag{10}$$

$$h^2(Z,K^{-mk})=0 \quad \text{for all} \quad m\geqq 1\,. \tag{11}$$

Before we apply this observation to a Bochner type argument, we should remark that in any conformal class on X, we can always choose a metric with constant scalar curvature. This is a direct consequence of Schoen and Aubin's work on the Yamabe problem [12]. In the following argument, we shall choose our self-dual metric to have constant scalar curvature.

Recall that we have the Dirac operator D_n and the twistor operator $\bar{D}_n$ on S^n_-:

$$D_n: S^n_- \to S^{n-1}_- \otimes S_+\,,$$

$$\bar{D}_n: S^n_- \to S^{n+1}_- \otimes S_+\,.$$

Let us choose inclusions

$$S^{n-1}_- \otimes S_+ \to S^n_- \otimes S_- \otimes S_+$$

and

$$S^{n+1}_- \otimes S_+ \to S^n_- \otimes S_- \otimes S_+\,.$$

As a result of Schur's lemma and the irreducibility of these bundles [2], these inclusions are uniquely defined up to a constant. Then there are universal nonzero constants λ_1 and λ_2 such that

$$Ds=\lambda_1 D_n s+\lambda_2 \bar{D}_n s\,,$$

where s is a smooth section of S^n_- and D is the connection on S^n_-. Since the splitting of

$$S^n_-\otimes(S_-\otimes S_+)=(S^{n-1}_-\otimes S_+)\otimes(S^{n+1}_-\otimes S_+)$$

is orthogonal with respect to the induced metric [1],

$$\|Ds\|^2=\lambda_1^2\|D_ns\|^2+\lambda_2^2\|\bar{D}_ns\|^2. \tag{12}$$

For the Dirac operator D_n, we have the well-known Weitzenböch formula [1, 7] that

$$D_n^*D_n=D^*D+\tfrac{1}{12}(n+2)u,$$

where u is the scalar curvature. Therefore,

$$\|D_ns\|^2=\|Ds\|^2+\tfrac{1}{12}(n+2)u\,\|s\|^2. \tag{13}$$

Then (12) can be expressed as

$$\lambda_2^2\|\bar{D}_ns\|^2=(1-\lambda_1^2)\,\|Ds\|^2-\tfrac{1}{12}(n+2)u\lambda_1^2\|s\|^2.$$

Note that $1-\lambda_1^2$ must be strictly positive, for otherwise, on a self-dual manifold with zero scalar curvature, e.g. torus or the $K3$-surface with the Calabi-Yau metric, we would have

$$\lambda_2^2\|\bar{D}_ns\|^2=(1-\lambda_1^2)\,\|Ds\|^2\leqq 0.$$

It would imply that all sections of S^n_- over a $K3$-surface is in the kernel of $\bar{D}_n$. This is a contradiction to the fact that $\bar{D}_n$ has only finite dimensional when the underlying self-dual 4-fold is compact [7].

Therefore, when u is negative,

$$\|\bar{D}_ns\|^2=a_1\|Ds\|^2+a_2(n+2)\,\|s\|^2$$

for some positive numbers a_1, a_2. Therefore, a section of S^n_- is in the kernel of $\bar{D}_n$ if and only if it is the zero section. By Eq. (2)

$$0=\dim\ker\bar{D}_n=h^0(Z,K^{-n/4})$$

for all $n>0$. In this case, the twistor space Z cannot fulfill the requirement of Eq. (10).

When the scalar curvature is zero, then Eqs. (12) and (13) become

$$\|D_ns\|^2=\|Ds\|^2,\qquad \|\bar{D}_ns\|^2=a_1\|Ds\|^2,$$

where a_1 is a positive constant. In particular,

$$\dim\ker\bar{D}_n=\dim\ker D_n,\quad \text{for all } n.$$

From Eqs. (1) and (2), we have

$$h^0(Z,K^{-n/4})=h^1(Z,K^{n/4+\frac{1}{2}}).$$

Then, by Serre duality,

$$\begin{aligned}h^0(Z,K^{-n/4})&=h^2(Z,K^{1-n/4-\frac{1}{2}})\\&=h^2(Z,K^{-(n-2)/4}).\end{aligned}$$

It implies that not both of Eqs. (10) and (11) can hold. The remaining possibility is that the scalar curvature u is positive.

Acknowledgements. The author thanks C. Lebrun for his initiation on the subject in this article. The author also thanks the faculty and staffs of the Mathematical Institute of Odense University where he finished this paper and the referee for his/her useful comments.

Note added in proof. Recently, M. Ville in Nancy also obtained results in a similar direction.

References

1. Atiyah, M.F., Hitchin, N.J., Singer, I.M.: Self-duality in four dimensional Riemannian geometry. Proc. R. Soc. Lond., Ser. A **362**, 425–461 (1978)
2. Besse, A.: Einstein manifolds. Berlin Heidelberg New York: Springer 1987
3. Donaldson, S.K., Friedman, R.: Private communication
4. Godement, R.: Topologie algébrique et théorie des Faisceaux. Paris: Hermann 1958
5. Grauert, H.: Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen. Publ. Math., Inst. Hautes Etud. Sci. **5**, 223–292 (1960)
6. Hartshorne, R.: Algebraic geometry. Graduate Texts in Math., Vol. 52). Berlin Heidelberg New York: Springer 1977
7. Hitchin, N.J.: Linear field equations on self-dual spaces. Proc. R. Soc. Lond., Ser. A **370**, 173–191 (1980)
8. Hitchin, N.J.: Kählerian twistor spaces. Proc. Lond. Math. Soc. (3) **43**, 133–150 (1981)
9. Moishézon, B.G.: On n-dimensional compact varieties with n algebraically independent meromorphic functions. Am. Math. Soc. Transl. **63**, 51–177 (1967)
10. Poon, Y.S.: Compact self-dual manifolds with positive scalar curvature. J. Differ. Geom. **24**, 97–132 (1986)
11. Poon, Y.S.: Small resolution of double solids as twistor spaces. J. Differ. Geom. (in press)
12. Schoen, R.: Conformal deformation of a Riemannian metric to construct scalar curvature. J. Differ. Geom. **20**, 479–495 (1984)
13. Ueno, K.: Classification theory of algebraic varieties and compact complex spaces. (Lecture Notes Math., Vol. 439). Berlin Heidelberg New York: Springer 1975

Received September 26, 1987; in revised form March 1, 1988