EINSTEIN-WEYL DEFORMATIONS AND SUBMANIFOLDS

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Motivated by new explicit positive Ricci curvature metrics on the four-sphere which are also Einstein–Weyl, we show that the dimension of the Einstein–Weyl moduli near certain Einstein metrics is bounded by the rank of the isometry group and that any Weyl manifold can be embedded as a hypersurface with prescribed second fundamental form in some Einstein–Weyl space. Closed four-dimensional Einstein–Weyl manifolds are proved to be absolute minima of the L^2-norm of the curvature of Weyl manifolds and a local version of the Lafontaine inequality is obtained. The above metrics on the four-sphere are shown to contain minimal hypersurfaces isometric to $S^1 \times S^2$ whose second fundamental form has constant length.

1. Introduction

Manifolds M^n with conformal structure $[g]$ and torsion-free affine connection D, such that parallel translation induces conformal transformations, are called *Weyl manifolds*. If, furthermore, the trace-free symmetric part of the Ricci curvature of D vanishes, the geometry is said to be *Einstein–Weyl*. Many examples and

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general results have been obtained [6, 17, 18, 19, 16, 22], in particular in dimensions three and four. In Sec. 2, we prove that closed four-dimensional Einstein–Weyl manifolds are minima for a natural functional. Furthermore, we observe that the Lafontaine inequality for scalar-flat half-conformally-flat four-manifolds has a local interpretation.

Recently, compact four-dimensional Einstein–Weyl manifolds with symmetry group of dimension at least four have been classified and this classification includes new explicit families of Einstein–Weyl structures on S^4 and $S^2 \times S^2$ containing the canonical Einstein metrics [15, 14]. In Secs. 3 and 4 we investigate the Einstein–Weyl moduli near Einstein metrics. We first show that these moduli are finite-dimensional and in certain cases show that the rank of the isometry group of the Einstein metric provides an upper bound for the dimension. Section 4 concludes by discussing some explicit families of Einstein–Weyl structures near Einstein metrics.

In Secs. 5 and 6 we study Weyl submanifolds. We show that the notion of second fundamental form extends naturally to Weyl geometry and that Weyl manifolds may be embedded as hypersurfaces with prescribed second fundamental form in Einstein–Weyl spaces. This extends known results for Einstein metrics proved by Koiso [9].

The techniques of the four-dimensional classification referred to above also gives new Einstein–Weyl geometries on S^n, $\mathbb{C}P(m)$ and the total space of the bundle $\mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O}) \to \mathbb{C}P(m-1)$. These conformal classes contain metrics of positive scalar curvature which happen to have positive Ricci curvature when the dimension is four. In Sec. 7, we give examples of minimal hypersurfaces in these manifolds and contrast their properties with examples given by Chern et al. [2].

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2. The Curvature Functional

The problem of finding metrics which are minima of some functional plays a central rôle in Riemannian geometry. One of the most natural such problems in dimension four is to find critical points of the functional

$$\mathcal{R}(g) = \int_M |R^g|^2 \operatorname{vol}_g$$

on the space of smooth Riemannian metrics of a given smooth, compact oriented four-manifold M. Here R^g is the Riemannian curvature tensor of the metric g. It is well known that the vanishing of the trace-free part of the Ricci curvature of the metric g (i.e. that g be an Einstein metric) is a sufficient condition for the metric to be an absolute minimum of $\mathcal{R}$ [12].

We may consider similar ideas in Weyl geometry. The compatibility between D and $[g]$ means that $Dg = \omega \otimes g$ for some one-form ω, depending on g, where g is a representative metric in the conformal class. We may express the *Weyl connection* D in terms of the metric connection ∇ of g using this one-form

$$D = \nabla + \frac{1}{2}\left(\omega^\sharp \otimes g - \mathrm{Id} \otimes \omega - \omega \otimes \mathrm{Id}\right) .$$

Note that if ω is exact, $\omega = d\lambda$, then D is just the metric connection of $e^{-\lambda} g$, cf.[18, 6].

Likewise, the curvature tensor R^D of D is given by

$$\begin{aligned} R^D = R^g &+ \frac{1}{2} d\omega \otimes g + \frac{1}{2} S(\nabla\omega) \owedge g \\ &+ \frac{1}{4} d\omega \owedge g + \frac{1}{4}\left(\omega \otimes \omega\right) \owedge g - \frac{1}{8}|\omega|^2 g \owedge g , \end{aligned} \tag{2.1}$$

where $S(\nabla\omega)(x,y) = \frac{1}{2}\left((\nabla_x\omega)(y) + (\nabla_y\omega)(x)\right)$ and for a two-tensor α, $\alpha \owedge g$ is the four-tensor

$$(\alpha \owedge g)(x,y,z,t) = \alpha(x,z)g(y,t) + \alpha(y,t)g(x,z) - \alpha(x,t)g(y,z) - \alpha(y,z)g(x,t) .$$

For symmetric α, this is just the Kulkarni–Nomizu product of α and g [1]. If r^g and r^D denote the Ricci curvatures of ∇ and D respectively and s^g, s^D their g-traces, then we write the decomposition into irreducible components

$$R^D = W + \frac{1}{n-2} S_0(r^D) \owedge g + \frac{1}{2n(n-1)} s^D g \owedge g + (\frac{1}{4} d\omega \owedge g + \frac{1}{2} d\omega \otimes g) ,$$

or equivalently

$$\begin{aligned} R^D = W &+ \left[\frac{1}{n-2} r_0^g + \frac{1}{2} S_0(\nabla\omega) + \frac{1}{4}\omega \otimes_0 \omega\right] \owedge g \\ &+ \left[\frac{1}{2n(n-1)} s^g - \frac{(n-2)}{8n}|\omega|^2 - \frac{1}{2n} d^*\omega\right] g \owedge g + (\frac{1}{4} d\omega \owedge g + \frac{1}{2} d\omega \otimes g) , \end{aligned}$$

where r_0^g, S_0, $\otimes_0$ indicate trace-free parts and we have used the relations

$$\begin{aligned} s^D &= s^g - \frac{1}{4}(n-2)(n-1)|\omega|^2 - (n-1)\, d^*\omega, \\ S(r^D) &= r^g + \frac{1}{4}(n-2)\left(\omega \otimes \omega - |\omega|^2 g + 2\nabla\omega - d\omega\right) - \frac{1}{2} d^*\omega\, g . \end{aligned}$$

Note that the $(3,1)$-tensor corresponding to R^D is conformally invariant.

As in the Riemannian case we look for extrema of the L^2-norm of the total curvature. In dimension four we have the well-defined functional

$$\mathcal{C}([g], D) = \int_M \left|R^D\right|_\otimes^2 \mathrm{vol}_g$$

on the space of smooth Weyl structures $([g], D)$ of a given smooth, compact oriented four-manifold M, where $|\cdot|_\otimes^2$ is the norm induced by g on $(3,1)$-tensors. Recall that the Euler characteristic $\chi(M)$ of M can be expressed in terms of the curvature of an arbitrary Riemannian metric g via the generalised Chern–Weil formula

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(|W_+|^2 + |W_-|^2 + \frac{1}{24}(s^g)^2 - \frac{1}{2}\left|r_0^g\right|^2\right)\mathrm{vol}_g,$$

where W_+ and W_- denote the self-dual and anti-self-dual Weyl curvatures respectively. Using this, it follows that

$$\mathcal{C}([g],D) = \int_M \left(4\left|S_0(r^D)\right|^2 + 2|d\omega|^2\right)\mathrm{vol}_g + 32\pi^2\chi(M),$$

where $|d\omega|^2 = \sum_{i<j} d\omega(e_i,e_j)^2$ for an orthonormal basis $\{e_1,\dots,e_4\}$ of TM. Thus, if $([g],D)$ is closed and satisfies the Einstein–Weyl equations $S_0(r^D) = 0$, the pair is an absolute minimum of $\mathcal{C}$. This new result may serve as further motivation for the study of the Einstein–Weyl equations

$$S(r^D) = \frac{1}{n}s^D g\,.$$

However, our examples of Einstein–Weyl structures on S^4 given later show that non-closed structures are not necessarily absolute minima for this functional.

Remark 2.1. If we express the signature $\tau(M)$ in terms of curvature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right)\mathrm{vol}_g,$$

then we may write the functional as

$$\mathcal{C}([g],D) = \int_M \left(16\,|W_\pm|^2 + 4\,|d\omega|^2 + \frac{1}{3}(s^D)^2\right)\mathrm{vol}_g - 32\pi^2\left(\chi(M) \pm 3\tau(M)\right).$$

Thus, Weyl-scalar-flat ($s^D = 0$), half-conformally-flat (i.e. either $W_+ = 0$ or $W_- = 0$) locally-conformally-metric ($d\omega = 0$) structures are also absolute minima of $\mathcal{C}([g],D)$. On the other hand, for Einstein–Weyl manifolds ($S_0 r^D = 0$), we have a Hitchin–Thorpe inequality

$$\chi(M) \geqslant \frac{3}{2}|\tau(M)| + \frac{1}{32\pi^2}\int_M |d\omega|^2\,\mathrm{vol}_g\,,$$

with equality if and only if M is half-conformally-flat and $s^D = 0$ [16]. Furthermore, for half-conformally-flat Weyl spaces with $s^D = 0$ we get a "local version" of the Lafontaine inequality [11] in this conformal setting

$$\chi(M) \leqslant \frac{3}{2}|\tau(M)| + \frac{1}{32\pi^2}\int_M |d\omega|^2\,\mathrm{vol}_g\,. \tag{2.2}$$

The conditions $s^D = 0$ and $d\omega = 0$ imply that g is locally conformal to a scalar-flat metric and thus the Lafontaine inequality is true also for locally-conformally scalar-flat half-conformally-flat manifolds. One has equality in (2.2) if and only if M is also Einstein–Weyl.

3. Slices

We have seen how a Weyl manifold is described by a pair (g, ω) consisting of a metric and a one-form. For a conformally equivalent metric $e^f g$ the corresponding one-form is $\omega + df$ and the new scalar curvature is $e^{-f} s^D$. A compact Weyl manifold M has a unique, up to homothety, metric g in the conformal class such that the one-form ω is co-closed [5]. We call this metric the *Gauduchon metric*. If furthermore the manifold is Einstein–Weyl, then the corresponding vector field $\omega^\sharp$ is Killing [22] and we have the *Gauduchon constant* G [17, 6] given by

$$\begin{aligned} G &= s^g - \frac{1}{4}(n+2)|\omega|^2 \\ &= s^D + \frac{1}{4}n(n-4)|\omega|^2 . \end{aligned}$$

Also, in this gauge the Einstein–Weyl equations for the pair (g, ω) become

$$r^g = \frac{1}{n} s^D g + \frac{1}{4}(n-2)(|\omega|^2 g - \omega \otimes \omega)$$

and for $n = 3$ the compact solutions to these equations are all known [22, 19]. For $n \geqslant 4$ it is easily seen [6, 19], that an Einstein–Weyl solution with $G \leqslant 0$ is either Einstein or belongs to a known one-parameter family of four-dimensional manifolds of type $S^1 \times S^3$, cf. Remark/Example (7) in Sec. 4. Thus, from the point of view of Einstein–Weyl geometry we may concentrate on the case of positive Gauduchon constant and scale the Gauduchon metric by a homothety to get $G = 1$.

The group of diffeomorphisms acts on the pair (g, ω) by pullback and $\varphi^*(d^{*_g}\omega) = d^{*_{\varphi^* g}}\varphi^*\omega$, that is, the pullback of the co-differential of ω with respect to g is the co-differential of the pullback of ω with respect to the pullback of g. Therefore, the conformal slice $d^*\omega = 0$ is invariant under the group of diffeomorphisms and we may think of the moduli space of Einstein–Weyl structures on M as a subset of the quotient space of $\mathcal{W} = \{(g, \omega) : d^*\omega = 0,\ G = 1\}$ by the action of the diffeomorphism group $\mathcal{D}$. Recall that the Ebin slice $\mathcal{S}(g)$ to the action of $\mathcal{D}$ at g is given infinitesimally by $\nabla^* \dot{g} = 0$, where $\dot{g} = \partial g_t / \partial t|_{t=0}$ is the tangent vector of a curve of metrics through $g_0 = g$ [3].

Definition 3.1. Let (g_0, ω_0) be an Einstein–Weyl solution on M. The subset $\widetilde{\mathcal{M}}$ of $\mathcal{S}(g_0) \times \Omega^1 M$ given by $d^*\omega = 0$, $G = 1$ and $r^g = \frac{1}{n} s^D g + \frac{1}{4}(n-2)(|\omega|^2 g - \omega \otimes \omega)$ is called the *premoduli space* of Einstein–Weyl structures around (g_0, ω_0).

To get a neighbourhood of (g_0, ω_0) in the *moduli space* $\mathcal{M}$ we must still divide by the action of the compact group of isometries of g_0 preserving ω_0.

4. Infinitesimal Deformations Around Einstein Metrics

Suppose (g_t, ω_t) is a smooth curve in the premoduli space around $(g_0, \omega_0) = (g, 0)$, where g_0 is an Einstein metric with $s^{g_0} = G = 1$. From the Killing condition $\nabla^t \omega_t = \frac{1}{2} d\omega_t$ we get $\nabla \dot{\omega} = \frac{1}{2} d\dot{\omega}$, so $\dot{\omega}^\sharp$ is a Killing vector for g_0, where $\dot{\omega} =$

$\partial\omega_t/\partial t|_{t=0}$. Furthermore we have $\dot{G} = 0$, so

$$\dot{s}^g = \frac{1}{4}(n+2)\big(2g(\omega,\dot{\omega}) - g(\dot{g}, \omega\otimes\omega)\big) = 0\,,$$

and similarly $\dot{s}^D = 0$.

Now, an arbitrary infinitesimal deformation satisfies

$$\dot{s}^g = \Delta \mathrm{Tr}\ \dot{g} + \nabla^*\nabla^*\dot{g} - g(r^g, \dot{g})\,,$$

so the condition $\dot{s}^g = 0$, the Ebin gauge condition $\nabla^*\dot{g} = 0$ and the fact that $r^{g_0} = \frac{1}{n}s^{g_0}g_0$ imply

$$\Delta \mathrm{Tr}\ \dot{g} = \frac{1}{n}s^{g_0}\mathrm{Tr}\ \dot{g}\,.$$

It then follows that $\mathrm{Tr}\dot{g} = 0$, because the smallest non-zero eigenvalue of the Laplacian is not less than $\frac{1}{n-1}s^{g_0}$ [13]. We now get

Theorem 4.1. *The metric part of the linearised Einstein–Weyl equations near an Einstein metric g_0 with $s^{g_0} = 1$ coincide with the linearised Einstein equations.*

Proof. We have

$$\begin{aligned}\dot{r} &= \frac{1}{n}(\dot{s}^D g + s^D\dot{g}) \\ &\quad + \frac{1}{4}(n-2)\left(|\omega|^2\dot{g} + 2g(\omega,\dot{\omega})g - g(\dot{g},\omega\otimes\omega)g - \dot{\omega}\otimes\omega - \omega\otimes\dot{\omega}\right) \\ &= \frac{1}{n}\dot{g}\,,\end{aligned}$$

since $\omega = 0$, $\dot{s}^D = 0$ and $s^D = 1$ at $t = 0$. □

Corollary 4.2. *The space of infinitesimal Einstein–Weyl deformations of an Einstein metric g_0 with $s^{g_0} = 1$ is finite-dimensional. In particular, let (M, g_0) be a locally symmetric Einstein manifold of compact type and let $\prod_{a=1}^N M_a$ be the irreducible decomposition of the universal Riemannian covering manifold $\widetilde{M}$. Consider the following lists of compact symmetric manifolds:*

(1) $\dfrac{SU(p+q)}{S(U(p)\times U(q))}$ $(p \geqslant q \geqslant 2)$, $\dfrac{E_6}{F_4}$,
$\dfrac{SU(\ell)}{SO(\ell)}$, $\dfrac{SU(2\ell)}{Sp(\ell)}$, $SU(\ell)$ $(\ell \geqslant 3)$;

(2) $\dfrac{G_2}{SO(4)}$, G_2;

(3) *Hermitian symmetric spaces of real dimension at least four;*

(4) S^2.

If $N = 1$ and M_a is not on list (1) or $N = 2$ and M_a is not on lists (1)–(3) or $N = 3$ and M_a is not on the lists (1)–(4), then (M, g_0) has at most an m-dimensional family *of infinitesimal Einstein–Weyl deformations, where m is the rank of the isometry group of g_0.*

Proof. Except for the excluded spaces, the Einstein metric has no infinitesimal Einstein deformations [10]. Therefore we are left with the Killing vectors $\dot{\omega}^\sharp$ modulo isometries of g_0. In general, the finite-dimensionality follows because the linearised Einstein equations are elliptic: as $\mathrm{Tr}\ \dot{g} = 0 = \nabla^*\dot{g}$, the equation $\dot{r} - \frac{1}{n}\dot{g} = 0$ is equivalent to $\nabla^*\nabla\dot{g} - 2\overset{\circ}{R}(\dot{g}) = 0$ [1]. □

Remarks and Examples. (1) It follows in particular, that the number of infinitesimal Einstein–Weyl deformations of the standard n-sphere is equal to the rank $\lfloor (n+1)/2 \rfloor$ of $SO(n+1)$. We will now give some examples which show that at least some of these deformations may be integrated to give a non-trivial moduli space.

(2) Consider an odd-dimensional sphere and the Hopf fibration $\pi : S^{2n+1} \to \mathbb{C}P(n)$. On the sphere we consider metrics given by

$$g = g_B + x\,\sigma \otimes \sigma,$$

for $x \in \mathbb{R}$, where g_B is the Fubini study metric and σ is the connection one-form with horizontal spaces given by the orthogonal complement to the fibres of the Hopf fibration. For certain values of x this metric is Einstein. We may consider the standard Einstein metric corresponding to $x = x_0$, say. Let $\omega = y\,\sigma$, for $y \in \mathbb{R}$, and consider the Weyl structure (g, ω) on S^{2n+1}. Using the O'Neill formulae for Riemannian submersions, the Einstein–Weyl operator $E = S(r^D) - \Lambda\, g$ is given by

$$E = \left(\lambda_B - \frac{x}{x_0}(\lambda_B - \lambda) - \Lambda\right) g_B + \left(\frac{x^2}{x_0}\lambda + \frac{1}{4}(n-1)y^2 - x\Lambda\right) \sigma \otimes \sigma\,,$$

where λ and λ_B are the Einstein constants for S^{2n+1} and $\mathbb{C}P(n)$ respectively [18]. The condition $\mathrm{Tr}\ E = 0$ gives

$$\Lambda = \lambda\frac{x}{x_0} + \frac{(n-1)}{4(n+1)}\frac{y^2}{x} + \frac{n}{n+1}\lambda_B\left(1 - \frac{x}{x_0}\right).$$

Therefore, the operator is a function

$$E : \mathbb{R}^2 \to \mathbb{R} \subseteq S^2T^*S^{2n+1},$$

where

$$E(x,y) = \frac{\lambda_B}{n+1}\left(1 - \frac{x}{x_0}\right) - \frac{n-1}{4(n+1)}\frac{y^2}{x}.$$

As the gradient of this smooth function satisfies

$$dE(x_0, 0) = \left(-\frac{\lambda_B}{x_0(n+1)}, 0\right) \neq 0\,,$$

it follows from the Inverse Function Theorem that $E^{-1}(0)$ is a one-dimensional submanifold of $\mathbb{R}^2$ near $(x_0, 0)$. Indeed, $E^{-1}(0)$ is the ellipsoid

$$\frac{(x - \frac{1}{2}x_0)^2}{x_0^2/4} + \frac{y^2}{\lambda_B x_0/(n-1)} = 1$$

of Einstein–Weyl solutions, with $(x_0, 0)$ corresponding to the standard Einstein metric and $(0,0)$ a degenerate structure. A priori we could allow x and y to be functions in an appropriate Banach space, but it is easily seen that we do not get more solutions this way. However, the method can be generalised to Einstein metrics of positive scalar curvature on principal S^1-bundles over a compact Kähler–Einstein base manifolds.

(3) For the three-sphere the number of infinitesimal deformations is two and all these have been integrated [19], using a relation [7] between four-dimensional self-dual manifolds and three-dimensional Einstein–Weyl geometry.

(4) The r-dimensional family of Einstein–Weyl structures obtained in [18, Theorem 4.2] on r-torus bundles over products of m Kähler–Einstein manifolds for $r \leqslant m$, also provide examples where the rank of the isometry group agrees with the dimension of the space of known deformations: the Einstein–Weyl structures are close to the Einstein metrics found by Wang & Ziller [23] and if the base manifolds are chosen to be products of Kähler–Einstein manifolds without continuous families of isometries, such as for example those found by Tian & Yau [20, 21] on the blow-up of $\mathbb{C}P(2)$ in k points, for $4 \leqslant k \leqslant 8$ (see [4]), then the isometry group of the Einstein metric on the torus bundle is the torus T^r itself, which has rank r. Note however, that the Einstein metrics on these T^r-bundles are not known to be rigid, so the full moduli space of Einstein–Weyl deformations could be larger.

(5) In the process of classifying compact four-dimensional Einstein–Weyl structures with big symmetry, Madsen [14] found a one-parameter family of Einstein–Weyl structures on S^4 which may be written as

$$\begin{aligned} g &= \frac{A(A\cot A - 1)}{\cot A - y\cot(Ay)}\, dy^2 + \sin^2(yA)\, g_{S^2} \\ &\quad + \frac{4A(A\cot A - 1)(\cot A - y\cot(Ay))}{(A + A\cot^2 A - \cot A)^2}\, d\theta^2 , \\ \omega &= -\frac{4A(\cot A - y\cot(Ay))}{A + A\cot^2 A - \cot A}\, d\theta , \end{aligned}$$

where $(y, \theta) \in (0,1) \times (0, 2\pi)$ are coordinates, g_{S^2} is the standard metric on S^2 and $A \in [0, \pi)$ is a parameter with $A = 0$ corresponding to the Einstein metric on S^4. This solution has $S^1 \times SO(3)$-symmetry and Madsen also found another one-parameter family with $U(2)$-symmetry, but we have not yet been able to find a two-parameter family of solutions integrating all the infinitesimal deformations on S^4.

(6) Corollary 4.2 also shows that $S^2 \times S^2$ can have at most a two-parameter family of Einstein–Weyl solutions near the Einstein metric and also in this case Madsen [14] found a new explicit one-parameter family of such structures.

(7) As an example of a moduli of Einstein–Weyl structures away from an Einstein metric, consider the manifold $S^1 \times S^{n-1}$. Let g_t be the product of the metric $t^2 d\theta^2$ on $S^1 = \{\exp(i\theta) : \theta \in [0, 2\pi)\}$ and the canonical metric g_{can} on S^{n-1} with sectional curvature one. Let ω_t be the one-form $2t\, d\theta$ and note that ω_t is harmonic with respect to g_t. This gives

$$\begin{aligned} r^{g_t} &= (n-2)g_{\mathrm{can}} = (n-2)(g_t - t^2 d\theta^2) \\ &= \frac{1}{4}(n-2)(|\omega_t|^2 g_t - \omega_t \otimes \omega_t) \end{aligned}$$

and hence (g_t, ω_t) is a one-parameter family of Einstein–Weyl geometries on $S^1 \times S^{n-1}$ with $s^{D_t} = 0$, which we call the *standard* structures. In four-dimensions this is the full local moduli: the Hitchin–Thorpe inequality for Einstein–Weyl manifolds in Remark 2.1, shows that any Einstein–Weyl structure on $S^1 \times S^3$ must satisfy $d\omega = 0$, but Gauduchon [6] proves that all such closed structures are standard.

5. Weyl Submanifolds

Let $(\overline{M}, [\bar{g}], \overline{D})$ be a Weyl manifold and $i\colon M \to \overline{M}$ an immersed submanifold and let π and $\hat{\pi}^{\perp}$ be the orthogonal projections from $i^*T\overline{M}$ to TM and the normal bundle $TM^{\perp}$, respectively. We pull back the conformal structure from $\overline{M}$ to M. In the following, X, Y, Z, W are vectors in TM and ξ belongs to $TM^{\perp}$. We obtain a torsion-free connection D on TM by $D_X Y = \pi(\overline{D}_X Y)$. This connection is compatible with the conformal structure. Indeed, if $\overline{D}\bar{g} = \bar{\omega} \otimes \bar{g}$, then $Dg = \omega \otimes g$, where $g = i^*\bar{g}$ and $\omega = i^*\bar{\omega}$. The conformal invariant β defined by the *Gauss formula*

$$\overline{D}_X Y = D_X Y + \beta(X, Y)$$

is called the *second fundamental form* of the Weyl structure. If α denotes the second fundamental form of the isometric immersion $i\colon (M, g) \to (\overline{M}, \bar{g})$, then $\beta = \alpha + \frac{1}{2}\pi^{\perp}(\bar{\omega}^{\sharp})g$. We also have a Weyl version of the *Weingarten formula*

$$\overline{D}_X \xi = -B_\xi X + D^N_X \xi\,.$$

Here $\bar{g}(\beta(X, Y), \xi) = g(B_\xi X, Y)$ and D^N becomes a connection on the normal bundle satisfying $D^N g^{\perp} = \omega \otimes g^{\perp}$, where $g^{\perp}$ is the metric on $TM^{\perp}$ induced by $\bar{g}$. The endomorphism B_ξ of TM is symmetric and the normal vector H which satisfies $\bar{g}(H, \xi) = \frac{1}{n}\mathrm{Tr} B_\xi$, $n = \dim M$, is called the *mean curvature* with respect to g. We shall focus on two conformally invariant conditions of the Weyl submanifold: $H = 0$ and $\beta = 0$, referred to as *minimal* and *totally geodesic*, respectively.

In the following we assume that M has co-dimension one. We choose a metric $\bar{g}$ and a normal ξ of unit length. Then the Gauss and Weingarten formulae become

$$\overline{D}_X Y = D_X Y + b(X, Y)\xi, \quad \overline{D}_X \xi = -BX - \frac{1}{2}\omega(X)\xi\,,$$

where $B = B_\xi$ and $\beta = b \otimes \xi$. For the curvature, we have the Gauss and Codazzi equations

$$\bar{g}(R^{\overline{D}}(X,Y)Z,W) = g(R^D(X,Y)Z,W) + b(Y,Z)b(X,W) - b(Y,W)b(X,Z),$$

$$\bar{g}(R^{\overline{D}}(X,Y)Z,\xi) = (D_Y b)(X,Z) - (D_X b)(Y,Z) + \frac{1}{2}\omega(X)b(Y,Z) - \frac{1}{2}\omega(Y)b(X,Z).$$

Using (2.1), the Ricci curvature can now be expressed as follows

$$r^{\overline{D}}(X,Y) = r^D(X,Y) + b^2(X,Y) - nh\, b(X,Y) + \varepsilon(X,Y),$$

$$r^{\overline{D}}(X,\xi) = n\, dh(X) + D^* b(X) + \tfrac{n}{2} h\,\omega(X) + \frac{1}{2} b(X,\omega^\sharp) + \frac{1}{2} d\bar{\omega}(X,\xi),$$

$$r^{\overline{D}}(\xi,\xi) = \mathrm{Tr}\varepsilon,$$

where $H = h\,\xi$, $b^2(X,Y) = \mathrm{Tr}(b(X,\cdot)b(Y,\cdot))$ and $\varepsilon(X,Y) = \bar{g}(R^{\overline{D}}(X,\xi)Y,\xi)$.

Thus, if the ambient space satisfies the Einstein–Weyl equations $S(r^{\overline{D}}) = \frac{1}{n+1} s^{\overline{D}} \bar{g}$, then

$$S(r^D) + b^2 - nh\, b + S(\varepsilon) = \frac{1}{n+1} s^{\overline{D}} g, \tag{5.1}$$

$$n\, dh + D^* b + \frac{n}{2} h\omega + \frac{1}{2} b(\omega^\sharp,\cdot) + \frac{1}{4}(n-1) d\bar{\omega}(\xi,\cdot) = 0, \tag{5.2}$$

$$\mathrm{Tr}\varepsilon = \frac{1}{n+1} s^{\overline{D}}. \tag{5.3}$$

6. Weyl Hypersurfaces in Einstein–Weyl Manifolds

Following the ideas of Koiso [9] on hypersurfaces of Einstein manifolds, we proceed to study the problem of embedding Weyl manifolds as hypersurfaces in Einstein–Weyl spaces.

For each choice of metric $\bar{g}$ on the ambient space, we may consider the mappings $i : M \times \mathbb{R} \to \overline{M}$ and $i_t : M \to \overline{M}$ given by $i(x,t) = \exp_x t\xi$ and $i_t(x) = i(x,t)$, respectively. Here exp denotes the exponential map of the metric $\bar{g}$. Then there is an open set $\mathcal{U}$ of $M \times \mathbb{R}$ containing $M \times \{0\}$ such that $g_t = i_t^* \bar{g}$ is a Riemannian metric on $\{\, x \in M : (x,t) \in \mathcal{U} \,\}$. Locally we have $\bar{g} = g_t + dt^2$ and $\bar{\omega} = \omega_t + \rho\, dt$, where $\omega_t = i_t^* \bar{\omega}$ and $\rho = \bar{g}\,(\bar{\omega}, \partial/\partial t)$. It is convenient to use the following

Lemma 6.1. *There exists a metric $\bar{g}$ in the conformal structure of a Weyl manifold $\overline{M}$ such that near a given Weyl hypersurface M we have $\bar{\omega} = \omega_t$.*

Proof. Choose a metric $\bar{g}_0$ with corresponding one-form $\bar{\omega}_0$. Let V be a neighbourhood in $\overline{M}$, where $\bar{\omega}_0 = \omega_t + \rho dt$ and set $U = M \cap V$. Under a conformal change $\bar{g}_0 \mapsto (\exp f_V)\bar{g}$, we have $\bar{\omega}_0 \mapsto \omega_t + \rho\, dt + d_M f_V + (\partial f_V/\partial t) dt$ and we choose f_V such that $\partial f_V/\partial t = -\rho$. Assume $\overline{M}$ is covered with such neighbourhoods $(V_i)_{i \in I}$

where furthermore the induced covering $(U_i)_{i\in I}$ of M admits a partition $(\varphi_i)_{i\in I}$ of unity. Let $f = \sum_{i\in I} \varphi_i f_{V_i}$. Then, on each neighbourhood V_j, $j \in I$, we have

$$\frac{\partial f}{\partial t} = \sum_{i\in I} \varphi_i \frac{\partial f_{V_i}}{\partial t} = -\rho \sum_{i\in I} \varphi_i = -\rho\,.$$

Note that near M we have $\rho\, dt = \pi^{\perp}(\bar{\omega}_0)$ and $(\partial f/\partial t)dt = \pi^{\perp}(df)$, so these expressions do not depend on the choice of t. Now, extend the function f from this neighbourhood of M by using a bump function to obtain the metric $\bar{g}$ on $\overline{M}$. □

We are now ready to prove the main existence theorem in this chapter.

Theorem 6.2. *Let $(M,[g],D)$ be a real analytic Weyl manifold with an analytic symmetric bilinear form β taking values in a real line bundle L on M. Then, there is a germ unique Einstein–Weyl space $(\overline{M},[\bar{g}],\overline{D})$ in which $(M,[g],D)$ is embedded as a hypersurface with second fundamental form β.*

Proof. We choose an analytic metric g with corresponding one-form ω representing the Weyl geometry on M. Locally, we may choose a section ξ trivialising the bundle L. We look for metrics g_t and one-forms ω_t, defined locally on M for small $t \in \mathbb{R}$, such that $g_0 = g$ and $\omega_0 = \omega$. Furthermore, we want the metric $\bar{g} = g_t + dt^2$ and the one-form $\bar{\omega} = \omega_t + \rho\, dt$ to satisfy the Einstein–Weyl equations locally on $M \times \mathbb{R}$. From the lemma it follows that we may assume $\rho = 0$. Thus, the second fundamental form $\alpha = a\,\xi$ of the isometric immersion coincides with the second fundamental form $\beta = b\,\xi$ of the Weyl space.

Now, Koiso [9] proved the relations $g' = -2a$ and $g'' = 2a^2 - \varepsilon^g$, where $\varepsilon^g(X,Y) = \bar{g}(R^{\bar{g}}(X,\xi)Y,\xi)$, $g' = \partial g/\partial t$ and $g'' = \partial^2 g/\partial t^2$. From (2.1) we then get

$$S(\varepsilon) = \frac{1}{4}(g')^2 - \frac{1}{2}g'' - \frac{1}{4}|\omega|^2 g + \frac{1}{4}\omega\otimes\omega + \frac{1}{2}S(\nabla\omega)\,.$$

If we take the trace of (5.1) to find $\mathrm{Tr}g''$ and substitute into (5.3), we get the Einstein–Weyl equations

$$g'' = -\frac{2}{n+1}s^{\overline{D}}g + 2S(r^D) - \frac{1}{2}g'\mathrm{Tr}g' + (g')^2 + S(\nabla\omega) + \frac{1}{2}\omega\otimes\omega - \frac{1}{2}|\omega|^2 g\,, \quad (6.1)$$

$$\omega' = \frac{1}{n-1}\left(2d\mathrm{Tr}g' + 2D^*g' + \omega\mathrm{Tr}g' + g'(\omega^\sharp,\cdot)\right)\,, \quad (6.2)$$

$$\mathrm{Tr}(g')^2 - (\mathrm{Tr}g')^2 = \frac{4(n-1)}{n+1}s^{\overline{D}} - \frac{4}{n}s^D\,. \quad (6.3)$$

Note that these equations hold locally on $M \times \mathbb{R}$, where the different tensors and operations refer to g_t.

Now, solve (6.3) for $s^{\overline{D}}$ and substitute into (6.1). By Cauchy–Kovalewski's existence theorem we can solve (6.1) and (6.2) locally, given the real analytic initial data

$$\omega_0 = \omega, \quad g_0 = g \quad \text{and} \quad g'_0 = -2a.$$

Then the global theorem follows from the local uniqueness. □

By imposing assumptions on completeness, we may obtain global uniqueness. To prove this, we need the following

Lemma 6.3. *Let M be a simply-connected, connected manifold with Weyl structures $([g_1], D_1)$ and $([g_2], D_2)$, which are analytic and agree on some open set U. If D_1 and D_2 are complete, then $([g_1], D_1) = f^*([g_2], D_2)$ globally for some diffeomorphism f of M.*

Proof. By Corollary 6.2 in [8], there is an extension of the identity map on U to an affine diffeomorphism f of M. Choose analytic representatives g_1 and g_2 of the two conformal structures. Then $f^*g_2 = \lambda g_1$ on U for some function λ on U. Extend λ to a global function by $\lambda = \frac{1}{n}\mathrm{Tr}(f^*g_2)$, where the trace is taken with respect to g_1. Then λg_1 and f^*g_2 are analytic tensors that agree on an open set, so they agree globally. □

The global uniqueness now follows easily:

Theorem 6.4. *Let $(M, [g], D)$ be a real analytic hypersurface of a simply-connected, connected, complete Einstein–Weyl manifold $(\overline{M}, [\bar{g}], \overline{D})$. Then $(\overline{M}, [\bar{g}], \overline{D})$ is unique up to Weyl diffeomorphism.* □

7. Examples with Special Second Fundamental Form

In this chapter we study hypersurfaces of some new Einstein–Weyl manifolds $\overline{M}$ of cohomogeneity one [15, 14]. We present these solutions in the Gauduchon gauge. In this gauge, the examples also have $\bar{\omega}$ pointing along the principal orbits which are the hypersurfaces we consider. Therefore the second fundamental form is given by the Gauduchon metric so this chapter also gives new examples in Riemannian geometry. In particular, we are going to see examples in dimension four of totally geodesic or minimal submanifolds of new metrics of positive Ricci curvature.

Consider first $\overline{M}$ equal to S^{n+1} or $S^2 \times S^{n-1}$ with an Einstein–Weyl structure admitting $S^1 \times SO(n)$ acting as a group of symmetries such that the orbit space is a closed interval and the principal orbit is $M = S^1 \times S^{n-1}$. The Weyl structure $(\bar{g}, \bar{\omega})$ must have the form [15, 14].

$$\bar{g} = dt^2 + f(t)^2 d\theta^2 + h(t)^2 g_{\mathrm{can}}, \qquad \bar{\omega} = \lambda(t)\, d\theta\,,$$

where $t \in [0, \ell]$, θ is the arc length parameter of a circle of length 2π, g_{can} is the canonical metric of sectional curvature one on S^{n-1} and $f, h, \lambda \in C^\infty[0, \ell]$ are some functions. Note that in this gauge, the pulled back one-form ω on M coïncides with the restriction of $\bar{\omega}$, so the second fundamental form of the Weyl structure is given by $a(X, Y) = \bar{g}(\overline{\nabla}_X Y, \xi)$, where $\xi = \partial/\partial t$ and $X, Y \in TM$. If $X_1, \ldots, X_{n-1}$ denotes a local orthonormal basis of TS^{n-1} and $X_n = \partial/\partial\theta$, then in this basis the second fundamental form is $a = -\mathrm{diag}(hh', \ldots, hh', ff')$ and we find that the mean curvature is

$$H = -\frac{1}{n}\frac{d}{dt}\left(\log(fh^{n-1})\right)\xi\,.$$

Now, the case $\overline{M} = S^{n+1}$ corresponds to the boundary conditions:

$$f'(0) = h(0) = h''(0) = 0, \quad h'(0) = 1, \tag{7.1}$$

$$f(\ell) = f''(\ell) = h'(\ell) = 0, \quad f'(\ell) = -1 \quad \text{and} \quad \lambda(\ell) = \lambda'(\ell) = 0. \tag{7.2}$$

Since $\log(fh^{n-1})$ tends to $-\infty$ both for $t \to 0$ and for $t \to \ell$, there must exist a $t_0 \in (0, \ell)$ such that $H(t_0) = 0$, so the corresponding orbit $M = S^1 \times S^{n-1}$ is minimal. This generalises the Clifford torus in (S^3, g_{can}) (see also [2]).

The case $\overline{M} = S^2 \times S^{n-1}$ corresponds to the boundary conditions (7.2) at both endpoints, so the same argument as above gives the existence of a minimal $S^1 \times S^{n-1}$ in $S^2 \times S^{n-1}$.

In [15, 14] it was proved that it is possible to find f, h, λ satisfying the Einstein–Weyl equations. It easily follows from Sec. 3 that the scalar curvature $s^{\bar{g}}$ of the Gauduchon metric is positive and that in addition, for $\dim \overline{M} = 4$, the Ricci curvature $r^{\bar{g}}$ is positive. In particular, the explicit metrics on S^4 from Sec. 4 have $r^{\bar{g}} > 0$ and each contains a minimal $S^1 \times S^2$ as the principal orbit at $y = y_0$, where

$$y_0(3 \cot y_0 - \tan y_0) + 1 = 4A \cot A\,,$$

if $A < \pi/2$. If we rescale the metric $\bar{g}$ by a homothety, so that its volume agrees with that of the canonical metric, then for this minimal principal orbit, the length of the second fundamental form is

$$6\sqrt{3}\frac{(y_0 \cot y_0 - A \cot A)}{(1 - A \cot A)^{1/2}} \sin A \cot^2 y_0\,. \tag{7.3}$$

As $A \to 0$, and hence $\bar{g} \to g_{\mathrm{can}}$, this value approaches 3 in agreement with the results of Chern et al. [2]. For (S^4, g_{can}), any hypersurface with $|a|^2$ non-zero and constant has $|a|^2 \geqslant 3$. However, for A small but non-zero, (7.3) is strictly less than 3. In fact, numerical calculations show that (7.3) is monotone decreasing and tends to 0 as A tends to π.

Next, let $\overline{M}$ be one of the following manifolds: S^{2m}, $\mathbb{C}P(m)$ or the total space of the $\mathbb{C}P(1)$-bundle $\mathbb{P}(\mathcal{O}(k) \oplus \mathcal{O}) \to \mathbb{C}P(m-1)$. In each case we consider a $U(m)$-symmetric Einstein–Weyl structure $(\bar{g}, \bar{\omega})$ where

$$\bar{g} = dt^2 + f(t)^2 \sigma^2 + h(t)^2 g_{\mathrm{FS}}, \qquad \bar{\omega} = \lambda(t)\, \sigma\,.$$

Here $t \in [0, \ell]$, $f, h, \lambda \in C^\infty[0, \ell]$, σ is the principal connection one-form of the Hopf fibration $S^{2m-1} \to \mathbb{C}P(m-1)$ and g_{FS} is the Fubini–Study metric on $\mathbb{C}P(m-1)$. The principal orbits are: $M = S^{2m-1}$, for $\overline{M} = S^{2m}$ and $\mathbb{C}P(m)$; and $M = S^{2m}/\mathbb{Z}_k$, for $\overline{M} = P(\mathcal{O}(k) \oplus \mathcal{O})$. The arguments in the $S^1 \times SO(n)$-symmetric case give, mutatis mutandis, Einstein–Weyl solutions $(\bar{g}, \bar{\omega})$ with particular principal orbits M as minimal submanifolds in the Gauduchon metric. In the case of $\overline{M} = S^{2m}$, the reflection around $t = \ell/2$ is actually an isometry and the corresponding S^{2m-1} is therefore totally geodesic. For instance, S^4 has a one-parameter family of $U(2)$-symmetric Einstein–Weyl structures different from the $S^1 \times SO(3)$-symmetric structures. Each of the $U(2)$-symmetric Gauduchon metrics therefore has $r^{\bar{g}} > 0$ and contains a totally geodesic equator. A similar totally geodesic example is obtained by embedding $S^1 \times S^n$ in the Einstein–Weyl manifold $S^1 \times S^{n+1}$.

Finally, we want to remark that even if all the principal orbits above are Einstein–Weyl, the induced structure on the orbits from the Einstein–Weyl geometry on $\overline{M}$ is rarely Einstein-Weyl. For example, no principal orbit $S^1 \times S^2$ or S^3 in the $S^1 \times SO(3)$ and $U(2)$-invariant Einstein–Weyl structures on S^4 are Einstein–Weyl. On the other hand, for the standard Einstein–Weyl structure on $S^1 \times S^n$, the embedded $S^n = \{*\} \times S^n$ is (Weyl) totally umbilic and Einstein.

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