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Inhomogeneous hypercomplex structures on homogeneous manifolds

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Abstract. We study deformations of hypercomplex structures on compact homogeneous spaces through the complex deformation theory of the associated twistor spaces. In general, we find complete parameter spaces of hypercomplex structures associated to compact semi-simple Lie groups. In particular, we discover the moduli space of hypercomplex structures on products of Hopf surfaces and on compact associated bundles of quaternionic projective spaces.

Introduction

A manifold is said to have a hypercomplex structure if there exist three integrable complex structures $\{I_1, I_2, I_3\}$ satisfying the identities of pure quaternions: $I_1 I_2 = -I_2 I_1 = I_3$. Hypercomplex manifolds interest physicists as they arise in the theory of $N = 4$ super-symmetric models with Wess-Zumino terms. In this context, Spindel, Sevrin, Troot and Van Proeyen discovered left-invariant hypercomplex structures on compact Lie groups [19]. When Joyce constructed homogeneous hypercomplex and homogeneous quaternionic manifolds, he independently discovered these hypercomplex structures [11].

Joyce's construction of homogeneous hypercomplex structures has its roots in the work of Borel [1], Samelson [18] and Wang [20] about homogeneous complex manifolds. It is also tied to Wolf's construction of symmetric, quaternionic Kähler manifolds [21]. Suppose that G is a compact semi-simple Lie group of rank r. Joyce and Spindel et al. found that the group $T^{2n-r} \times G$ has a left-invariant hypercomplex structure, where n is the number of commuting $\mathfrak{sp}(1)$ subalgebras in the Lie algebra $\mathfrak{g}$ of G. Here T^{2n-r} is the $(2n-r)$-dimensional compact Abelian group. Joyce also finds that for appropriate choice of subgroups P and Abelian groups T^ℓ, the homogeneous space $T^\ell \times G/P$ has left invariant hypercomplex structures. In Section 1, we explain how the hypercomplex structures on $T^{2n-r} \times G$ are defined. We defer the discussion on general homogeneous spaces until Section 5. In this paper, we investigate deformations of these hypercomplex structures. With focus on group manifolds, the purpose of this paper is to construct inhomogeneous hyper-

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complex structures through deformations and to identify parameter spaces for all deformations of homogeneous hypercomplex structures.

Deformations of homogeneous complex manifolds were studied by Griffiths [8]. An important ingredient in his paper is a relation between the homogeneous spaces and flag manifolds. Applying twistor theory [14], [17], we establish a link between the twistor space of $T^{2n-r} \times G$ and flag manifolds. The twistor space associated to the hypercomplex manifold $T^{2n-r} \times G$ is a complex manifold W encompassing all complex structures of the given hypercomplex structure. There is a holomorphic map p from the twistor space W to $\mathbb{CP}^1$ so that the pre-images of points in $\mathbb{CP}^1$ are the complex structures of $T^{2n-r} \times G$ [15], [17]. It is known that deformations of hypercomplex structures are equivalent to deformations of the complex structure of W and the holomorphic map p [15]. The first step to describe the deformation theory is a computation of the relevant cohomology groups on the twistor spaces. It is done via a Leray's spectral sequence established by the link between the twistor spaces and flag manifolds. The computation is possible due to the vanishing theorems of Borel-Hirzebruch and Bott [2], [3]. From this computation, we find that the obstruction space to deformations is non-trivial. Therefore, in Section 4 we complete our calculation of the deformations through Kuranishi theory. The result is:

Theorem. *Suppose G is a compact semi-simple Lie group of rank r. Then the local moduli at a generic deformation of left-invariant hypercomplex structures on $T^{2n-r} \times G$ is a smooth manifold of dimension $n(n+r)$.*

The identity component of the group of hypercomplex symmetries of a generic deformation is the Abelian group T^{2n}.

With more details, this theorem is stated as Theorem 1 in Section 4. All these ideas can be applied to calculate local moduli at a generic deformation of left-invariant hypercomplex structures on homogeneous spaces with respect to compact semi-simple Lie groups. We highlight the main steps for such generalization in Section 5. The analogue result is Theorem 2.

In the computation of the Kuranishi family, it is seen that the infinitesimal deformations of hypercomplex structures on $T^{2n-r} \times G$ are entirely described by the infinitesimal deformations of an Abelian variety contained in the twistor space. Inspired by the theory of Abelian varieties, we construct a family of hypercomplex structures on $(S^3 \times S^1)^n$. This family is complete due to the theorem above. In fact, we find the global moduli space of hypercomplex structures on $(S^3 \times S^1)^n$ (near the left invariant structures). To be precise, we obtain

Theorem. *The quotient space $\big((\mathbb{R}/\mathbb{Z})^{n^2} \times \mathrm{GL}(n, \mathbb{R})\big)/\big(\mathbb{Z}_2^n \times \mathrm{GL}(n, \mathbb{Z})\big)$ is a complete moduli space for hypercomplex structures on the product manifold $(S^3 \times S^1)^n$.*

This is Theorem 3 proved in Section 6.1. The action of the group $\mathbb{Z}_2^n \times \mathrm{GL}(n, \mathbb{Z})$ on the space $(\mathbb{R}/\mathbb{Z})^{n^2} \times \mathrm{GL}(n, \mathbb{R})$ is given by (30) and Lemma 16.

To illustrate our theory for homogeneous spaces, we construct the moduli space of hypercomplex structures on the homogeneous space $S^1 \times \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$ in Section 6.4. The result is

Theorem. *The moduli space of hypercomplex structures on* $T^1 \times \dfrac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}$ *is the quotient space* $\mathbb{R}^+ \times \dfrac{U}{\mathscr{W}}$ *where* U *is a maximal torus of the group* $\mathrm{Sp}(n)$ *and* $\mathscr{W}$ *is the corresponding Weyl group.*

1. Hypercomplex structures on Lie groups

Suppose G is a compact semi-simple Lie group. Let U be a maximal torus. The space G/U has a left-invariant homogeneous complex structure [1], § 29. The (1, 0)-forms at the identity coset is the space of positive roots with respect to the Cartan subalgebra $\mathfrak{u}_{\mathbb{C}}$. The complex structures at the other points of the quotient space are obtained by left translations.

The Joyce construction is similar. It starts with the Wolf decomposition of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ [21]. Choose a system of ordered roots with respect to $\mathfrak{u}_{\mathbb{C}}$. Let α_1 be a maximal positive root, and $\mathfrak{h}_1$ the dual space of α_1. Let ∂_1 be the $\mathfrak{sp}(1)$-subalgebra of $\mathfrak{g}$ such that its complexification is isomorphic to $\mathfrak{h}_1 \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1}$ where $\mathfrak{g}_{\alpha_1}$ and $\mathfrak{g}_{-\alpha_1}$ are the root spaces for α_1 and $-\alpha_1$ respectively. Let $\mathfrak{b}_1$ be the centralizer of ∂_1. Then there is a vector subspace $\mathfrak{f}_1$ composed of root spaces such that $\mathfrak{g} = \mathfrak{b}_1 \oplus \partial_1 \oplus \mathfrak{f}_1$. If $\mathfrak{b}_1$ is not Abelian, Joyce applies this decomposition to it. By inductively searching for $\mathfrak{sp}(1)$ subalgebras, he finds the following [11], Lemma 4.1.

Lemma 1. *The Lie algebra* $\mathfrak{g}$ *of a compact Lie group* G *decomposes as*

$$\mathfrak{g} = \mathfrak{b} \bigoplus_{j=1}^{n} \partial_j \bigoplus_{j=1}^{n} \mathfrak{f}_j, \tag{1}$$

with the following properties. (1) $\mathfrak{b}$ *is Abelian and* ∂_j *is isomorphic to* $\mathfrak{sp}(1)$. (2) $\mathfrak{b} \bigoplus_{j=1}^{n} \partial_j$ *contains* $\mathfrak{u}$. (3) *Set* $\mathfrak{b}_0 = \mathfrak{g}$, $\mathfrak{b}_n = \mathfrak{b}$ *and* $\mathfrak{b}_k = \mathfrak{b} \bigoplus_{j=k+1}^{n} \partial_j \bigoplus_{j=k+1}^{n} \mathfrak{f}_j$. *Then* $[\mathfrak{b}_k, \partial_j] = 0$ *for* $k \geqq j$. (4) $[\partial_l, \mathfrak{f}_l] \subset \mathfrak{f}_l$. (5) *The adjoint representation of* ∂_l *on* $\mathfrak{f}_l$ *is reducible to a direct sum of the irreducible* 2*-dimensional representations of* $\mathfrak{sp}(1)$.

Note that this theorem is true for the Lie algebra of any compact Lie group. We refer to this decomposition of the algebra $\mathfrak{g}$ as the Joyce decomposition. In this decomposition n is the number of commuting $\mathfrak{sp}(1)$ subalgebras, and the rank of the algebra $\mathfrak{g}$ is equal to $n + \dim \mathfrak{b}$.

Proposition 1. *The Joyce decompositions of compact simple Lie algebras are*

	$\mathfrak{a}_{2\ell}$	$\mathfrak{a}_{2\ell-1}$	$\mathfrak{b}_\ell$	$\mathfrak{c}_\ell$	$\mathfrak{d}_{2\ell}$	$\mathfrak{d}_{2\ell+1}$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{f}_4$	$\mathfrak{g}_2$
n	ℓ	ℓ	ℓ	ℓ	2ℓ	2ℓ	4	7	8	4	2
$\dim b$	ℓ	$\ell-1$	0	0	0	1	2	0	0	0	0

Table: for $\ell \geqq 1$

Proof. We derive the decompositions through the following recursive relations:

$$\mathfrak{a}_\ell = \mathfrak{u}(1) \oplus \mathfrak{a}_{\ell-2} \oplus \partial_1 \oplus \mathfrak{f}_1, \quad \mathfrak{b}_\ell = \mathfrak{b}_{\ell-2} \oplus \partial_2 \oplus \partial_1 \oplus \mathfrak{f}_1,$$
$$\mathfrak{c}_\ell = \mathfrak{c}_{\ell-1} \oplus \partial_1 \oplus \mathfrak{f}_1, \quad \mathfrak{d}_\ell = \mathfrak{d}_{\ell-2} \oplus \partial_2 \oplus \partial_1 \oplus \mathfrak{f}_1,$$
$$\mathfrak{e}_6 = \mathfrak{a}_5 \oplus \partial_1 \oplus \mathfrak{f}_1, \quad \mathfrak{e}_7 = \mathfrak{d}_6 \oplus \partial_1 \oplus \mathfrak{f}_1, \quad \mathfrak{e}_8 = \mathfrak{e}_7 \oplus \partial_1 \oplus \mathfrak{f}_1,$$
$$\mathfrak{f}_4 = \mathfrak{c}_3 \oplus \partial_1 \oplus \mathfrak{f}_1, \quad \mathfrak{g}_2 = \partial_2 \oplus \partial_1 \oplus \mathfrak{f}_1.$$

q.e.d.

Let G be a compact semi-simple Lie group. Then

$$(2n-r)\,\mathfrak{u}(1) \oplus \mathfrak{g} \cong \mathbb{R}^n \bigoplus_{j=1}^{n} \partial_j \bigoplus_{j=1}^{n} \mathfrak{f}_j. \tag{2}$$

At the tangent space of the identity element of $T^{2n-r} \times G$, i.e. the Lie algebra $(2n-r)\,\mathfrak{u}(1) \oplus \mathfrak{g}$, a hypercomplex structure $\{I_1, I_2, I_3\}$ is defined as follows. Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Choose isomorphisms ϕ_j from $\mathfrak{sp}(1)$, the real vector space of imaginary quaternions, to ∂_j. It gives a real linear identification from the quaternions $\mathbb{H}$ to $\langle e_j \rangle \oplus \partial_j$. Define the action of I_a on $\mathfrak{f}_j$ by $I_a(v) = [v, \phi_j(\iota_a)]$ where $\iota_1 = i, \iota_2 = j, \iota_3 = k$. The complex structures $\{I_1, I_2, I_3\}$ at the other points of the group $T^{2n-r} \times G$ are obtained by left translations.

Due to Proposition 1, Joyce's construction for compact simple Lie groups yields the following lists of left-invariant hypercomplex manifolds.

Proposition 2. *The following groups have homogeneous hypercomplex structures*:

$$\mathrm{SU}(2\ell+1), \quad T^1 \times \mathrm{SU}(2\ell), \quad T^\ell \times \mathrm{SO}(2\ell+1), \quad T^\ell \times \mathrm{Sp}(\ell), \quad T^{2\ell} \times \mathrm{SO}(4\ell),$$
$$T^{2\ell} \times \mathrm{SO}(4\ell+2), \quad T^2 \times E_6, \quad T^7 \times E_7, \quad T^8 \times E_8, \quad T^4 \times F_4, \quad T^2 \times G_2.$$

This list of hypercomplex manifolds is known to Spindel et al. [19], Table 1.

1.1. Hypercomplex structures on SU(2) × U(1). The hypercomplex structures on the Hopf manifold $S^3 \times S^1$ are classified by Boyer and Kato [4], [12]. Given any complex number γ such that $0 < |\gamma| < 1$, the left multiplication by γ on the right module $\mathbb{H}$ generates an integer group Γ acting on $\mathbb{H}^*$. We choose spherical coordinates (q, ϱ) so that $\mathbb{H}^*$ is identified to the product space $S^3 \times \mathbb{R}^+$. For $\gamma = re^{2\pi i\theta}$, the action of γ^n is $(e^{2\pi i n\theta} q, r^n \varrho)$. Since left multiplication commutes with right multiplication of the quaternions, the quotient space $(S^3 \times \mathbb{R}^+)/\Gamma$ is hypercomplex. The map $(q, \varrho) \mapsto (e^{-2\pi i \frac{\theta \ln \varrho}{\ln r}} q, e^{2\pi i \frac{\ln \varrho}{\ln r}})$ from $S^3 \times \mathbb{R}^+$ to $S^3 \times S^1$ factors through the quotient space $(S^3 \times \mathbb{R}^+)/\Gamma$, and defines a diffeomorphism from $(S^3 \times \mathbb{R}^+)/\Gamma$ to the Hopf manifold $S^3 \times S^1$. Since the hypercomplex structures on the quotient generated by the group of γ and $\overline{\gamma}$ are equivalent, up to a $\mathbb{Z}_2$ action generated by complex conjugation, these hypercomplex structures on $S^3 \times S^1$ are parametrized by the punctured disk $\{\gamma \in \mathbb{C} : 0 < |\gamma| < 1\}$. We shall revisit this example in great details in Example 6.1.

Now we treat the manifold $S^3 \times S^1$ as the Lie group $\mathrm{SU}(2) \times \mathrm{U}(1)$. The group multiplication on $\mathrm{SU}(2) \times \mathrm{U}(1)$ is $(q_1, e^{2\pi i \ln \varrho_1}) \cdot (q_2, e^{2\pi i \ln \varrho_2}) = (q_1 q_2, e^{2\pi i (\ln \varrho_1 + \ln \varrho_2)})$. Therefore, it is covered by the multiplication on $S^3 \times \mathbb{R}^+$ defined by

$$(q_1, \varrho_1) \cdot (q_2, \varrho_2) = (q_1 q_2, \varrho_1 \varrho_2).$$

It follows that the left-invariant hypercomplex structures on the Lie group $\mathrm{SU}(2) \times \mathrm{U}(1)$ is parametrized by the space $\{-r : 0 < r < 1\} \cup \{r : 0 < r < 1\}$.

1.2. Hypercomplex structures on SO(3) × U(1). Given the action of $\mathbb{Z} \times \mathbb{Z}_2$ on $S^3 \times \mathbb{R}^+$ generated by γ above and $\varepsilon(q, \varrho) = (-q, \varrho)$, we consider the quotient $(S^3 \times \mathbb{R}^+)/(\Gamma \times \langle \varepsilon \rangle)$. As the adjoint representation Ad of $\mathrm{SU}(2)$ on its real algebra is a double covering from $\mathrm{SU}(2)$ onto $\mathrm{SO}(3)$, the map

$$(q, \varrho) \mapsto \left(\mathrm{Ad}(e^{-2\pi i \frac{\theta \ln \varrho}{\ln r}} q), e^{2\pi i \frac{\ln \varrho}{\ln r}}\right) \tag{3}$$

determines a diffeomorphism from $(S^3 \times \mathbb{R}^+)/(\Gamma \times \langle \varepsilon \rangle)$ to $\mathrm{SO}(3) \times \mathrm{U}(1)$.

Since the hypercomplex structures defined by the quotients with respect to $\langle \gamma \rangle \times \langle \varepsilon \rangle$ and $\langle -\gamma \rangle \times \langle \varepsilon \rangle$ are equivalent, they are parametrized by the space $\{re^{i\theta} : 0 < r < 1, -\pi/2 \leqq \theta \leqq \pi/2\}$, and the left invariant hypercomplex structures on $\mathrm{SO}(3) \times \mathrm{U}(1)$ are parametrized by the interval $\{r : 0 \leqq r \leqq 1\}$.

1.3. Hypercomplex structures on U(2). Although the group $\mathrm{U}(2)$ is not semi-simple, we include its description as a hypercomplex manifold with left-invariant hypercomplex structure for later exposition.

Considering S^3 as the space of unit quaternions $\{q = (w_1 + jw_2) \in \mathbb{H} : |w_1|^2 + |w_2|^2 = 1\}$, and $\mathrm{U}(1)$ as the space of unit complex numbers, we define a diffeomorphism from $S^3 \times S^1$ to the group $\mathrm{U}(2)$ by

$$(q, e^{i\theta}) \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} w_1 & -\bar{w}_2 \\ w_2 & \bar{w}_1 \end{pmatrix}. \tag{4}$$

The multiplication for $\mathrm{U}(2)$ in the coordinates $(q, e^{i\theta})$ is given by

$$(q_1, e^{i\theta_1}) \cdot (q_2, e^{i\theta_2}) = (e^{-i\frac{\theta_2}{2}} q_1 e^{i\frac{\theta_2}{2}} q_2, e^{i(\theta_1 + \theta_2)}).$$

The multiplication on $S^3 \times \mathbb{R}^+$ is

$$(q_1, \varrho_1) \cdot (q_2, \varrho_2) = (e^{-\pi i \ln \varrho_2} q_1 e^{\pi i \ln \varrho_2} q_2, \varrho_1 \varrho_2).$$

Note that the left multiplication is again hypercomplex. For any $\gamma = re^{2\pi i\theta}$, the action generated by $\gamma(q,\varrho) = (e^{2\pi i\theta}q, r\varrho)$ commutes with left multiplication if and only if $2\theta + \ln r$ is an integer. Thus the space of U(2)-left invariant hypercomplex structures is $\{-re^{-\pi i \ln r} : 0 < r < 1\} \cup \{re^{-\pi i \ln r} : 0 < r < 1\}$.

2. The twistor spaces of $T^{2n-r} \times G$

In this section, we describe the twistor space of the hypercomplex structure on $T^{2n-r} \times G$ and other related objects. We refer the reader to [15] for general twistor theory on hypercomplex manifolds. We focus on the algebraic nature of the twistor space of $T^{2n-r} \times G$.

By construction, the left action of the group $T^{2n-r} \times G$ on itself generates hypercomplex automorphisms, i.e. automorphisms which are holomorphic with respect to each complex structure of the hypercomplex structure. However, there is a subgroup in $T^{2n-r} \times G$ whose right action is also hypercomplex. When $\mathfrak{b}$ is not trivial, let B be the real Abelian subgroup of G whose algebra is $\mathfrak{b}$. As $\mathfrak{b}$ is in the commutator of all ∂_j, the adjoint action of $\mathfrak{b}$ preserves the hypercomplex structure on the components $\langle e_j \rangle \oplus \partial_j$, for $1 \leqq j \leqq n$. Since each $\mathfrak{f}_j$ is a direct sum of root spaces, and $\mathfrak{b}$ is contained in the Cartan subalgebra, the adjoint action of $\mathfrak{b}$ also leaves the hypercomplex structure on the component $\mathfrak{f}_j$ invariant. Since the hypercomplex structure is left invariant, it follows that the *right* action of the group B is hypercomplex.

Lemma 2. *The algebra of hyper-holomorphic vector fields on $T^{2n-r} \times G$ contains the direct sum of the Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$, where $\mathfrak{a}$ is the algebra of right-invariant vector fields generated by $T^{2n-r} \times B$.*

There is also a distinguished subgroup of right multiplication preserving the collection of complex structures although it does not preserve any individual one.

Lemma 3. *Let $\Delta : \mathfrak{sp}(1) \to \mathfrak{g}$ be defined by $\Delta(\iota_a) = \sum_{j=1}^{n} \phi_j(\iota_a)$, for $1 \leqq a \leqq 3$. Let Δ also represent the corresponding group homomorphism. Then for any complex structure $I_v = v_1 I_1 + v_2 I_2 + v_3 I_3 = (v_1, v_2, v_3)$ in the given hypercomplex structure, and any A in* Sp(1),

$$\mathrm{Ad}_{\Delta(A)} I_v \mathrm{Ad}_{\Delta(A)^{-1}} = I_{\mathrm{Ad}_A v}. \tag{5}$$

In particular, the right action of the subgroup $\Delta(\mathrm{Sp}(1))$ is a group of quaternionic transformations on $T^{2n-r} \times G$.

Proof. The algebra of endomorphisms on the manifold $T^{2n-r} \times G$ generated by I_1, I_2 and I_3 is isomorphic to $\mathfrak{sp}(1)$. We have $I_{\mathrm{Ad}_A v} = \mathrm{Ad}_A(I_v)$. It follows that the infinitesimal version of (5) is $[\mathrm{ad}\,\Delta(\iota), I_v] = [\iota, I_v]$, for any element ι in $\mathfrak{sp}(1)$. As this identity is linear in both the variable v and the variable ι, it suffices to verify that

$$[\mathrm{ad}\,\Delta(\iota_a), I_a] = 0 \quad \text{and} \quad [\mathrm{ad}\,\Delta(\iota_a), I_b] = 2I_c, \tag{6}$$

when (abc) is an even permutation of (123).

We verify the second identity, and leave the verification of the first identity to the reader. For any element w in the Lie algebra $(2n-r)\,\mathfrak{u}(1)_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}$,

$$[\operatorname{ad}\Delta(\iota_a), I_b]\,w = \sum_j \{[\phi_j(\iota_a), I_b w] - I_b[\phi_j(\iota_a), w]\}\,. \tag{7}$$

Recall that the Lie algebra for $T^{2n-r} \times G$ has a decomposition (2). Suppose that w is contained in $\langle e_k \rangle \oplus \partial_k$ for some k. The identity (7) is reduced to

$$\begin{aligned}
&[\phi_k(\iota_a), I_b w] - I_b[\phi_k(\iota_a), w] \\
&= [\phi_k(\iota_a), w\phi_k(\iota_b)] - [\phi_k(\iota_a), w]\,\phi_k(\iota_b) = 2w\phi_k(\iota_c) = 2I_c w\,.
\end{aligned}$$

Suppose that w is contained in $\mathfrak{k}_l$ for some l. Using the Jacobi identity, then the identity (7) is reduced to

$$\begin{aligned}
&[\phi_l(\iota_a), I_b w] - I_b[\phi_l(\iota_a), w] \\
&= [\phi_l(\iota_a), [w, \phi_l(\iota_b)]] - [[\phi_l(\iota_a), w], \phi_l(\iota_b)] \\
&= [w, [\phi_l(\iota_a), \phi_l(\iota_b)]] = 2[w, \phi_l(\iota_c)] = 2I_c w\,.
\end{aligned}$$

The verification of the second identity in (6) is now completed. q.e.d.

Let us choose the map ϕ_j so that the dual α_j of $\phi_j(\iota_1)$ is a positive root, and $\phi_j(\iota_2) \pm i\phi_j(\iota_3)$ spans the root spaces $\mathfrak{g}_{\pm\alpha_j}$. As observed by Joyce [11], Equation (10) and (11), the (1, 0)-form at the identity element with respect to I_1 on the complement of the Abelian part is precisely the direct sum of all positive root spaces. Since the Abelian part is the algebra of $T^{2n-r} \times U$, we have

Lemma 4. *With respect to the complex structure I_1, there is a holomorphic map from $T^{2n-r} \times G$ onto the flag manifold G/U.*

Through Lemma 3, we see that the above lemma can be applied to each complex structure of the hypercomplex structure. We shall explore this observation through twistor theory.

Let W be the twistors space of $T^{2n-r} \times G$. Since the Obata connection trivializes W as a fiber bundle over $T^{2n-r} \times G$, the space W is smoothly the product of $T^{2n-r} \times G$ and the 2-sphere S^2. To describe the complex structure on W, we recall that the 2-sphere S^2 parameterizes the complex structures of the hypercomplex structure on $T^{2n-r} \times G$. If $v = (v_1, v_2, v_3)$ is a unit vector in $\mathbb{R}^3$, we denote the complex structure $v_1 I_1 + v_2 I_2 + v_3 I_3$ by I_v. Let $\mu = (1, 0, 0)$. Consider S^2 as the quotient $\mathrm{Sp}(1)/\mathrm{U}(1)$ with $\mathrm{U}(1)$ the isotropy at μ. Let J_v be the standard complex structure at v on S^2 defined by the orientation $\{I_1, I_2, I_3\}$.

Let $(1, e)$ be the identity element in the group $T^{2n-r} \times G$. The complex structure $\mathscr{I}$ at the point $(1, e, v)$ in $T^{2n-r} \times G \times S^2$ is the direct sum $\mathscr{I}_{(1,e,v)} = (I_v, J_v)$. Note that I_v is a complex structure for $T^{2n-r} \times G$ at $(1, e)$. Let $A \in \mathrm{Sp}(1)$ such that $A\mu = v$. Define the action $\mathscr{A}$ of A on $T^{2n-r} \times G \times S^2$ by $\mathscr{A}(t, g, w) = \big(t, \Delta(A)\,g\,\Delta(A)^{-1}, Aw\big)$. By Lemma 3, $(I_v, J_v) = d\mathscr{A} \circ (I_1, J_\mu)$.

Since the complex structure at (t, g, v) is obtained by left translations of the complex structure $\mathscr{I}_{(1,e,v)}$ by the group element (t, g), we extend the $\mathscr{A}$-action to an action of $T^{2n-r} \times G \times \mathrm{Sp}(1)$ on $T^{2n-r} \times G \times S^2$ by

$$\mathscr{A}(s, h, A;\ t, g, B\mu) = \big(st, h\Delta(A)g\Delta(A)^{-1}, AB\mu\big). \tag{8}$$

It shows that the group of transformations is a semi-direct product of $T^{2n-r} \times G$ with $\mathrm{Sp}(1)$. We denote this group by $T^{2n-r} \times G \bowtie \mathrm{Sp}(1)$. We have just seen that the complex structure on the twistor space is homogeneous with respect to this group. Thus, we have

Lemma 5. *The twistor space W is the homogeneous complex manifold*

$$W = \frac{T^{2n-r} \times G \bowtie \mathrm{Sp}(1)}{\mathrm{U}(1)}.$$

In the semi-direct product, we are interested in several subgroups. Recall that U is a maximal torus in G. Let E be the product subgroup $T^{2n-r} \times U$ in $T^{2n-r} \times G$. Note that $T^{2n-r} \times B$ is a subgroup of E such that $\dim E = 2\dim(T^{2n-r} \times B)$.

The map from $\mathrm{Sp}(1)$ to $T^{2n-r} \times G \bowtie \mathrm{Sp}(1)$ defined by $\gamma(A) = \big(1, \Delta(A)^{-1}, A\big)$ is a group homomorphism. The subgroup $\gamma\big(\mathrm{Sp}(1)\big)$ commutes with the subgroup E. It follows that the subgroup generated by E and $\gamma\big(\mathrm{Sp}(1)\big)$ is a direct product. Abusing notation, we denote this product by $E \bowtie \mathrm{Sp}(1)$. Considering $\mathrm{U}(1)$ as a subgroup in $E \bowtie \mathrm{Sp}(1)$ through the embedding in $\mathrm{Sp}(1)$, we have a homogeneous manifold $F = \dfrac{E \bowtie \mathrm{Sp}(1)}{\mathrm{U}(1)}$.

Lemma 6. *There is a locally trivial holomorphic fibration Φ from W to the flag manifold $Z = G/U$ with F as fiber.*

Proof. We define a map ϕ from $T^{2n-r} \times G \bowtie \mathrm{Sp}(1)$ to G by

$$\phi(t, g, A) = g\Delta(A). \tag{9}$$

It is a group homomorphism. Since this map sends the $\mathrm{U}(1)$-subgroup of the $\mathrm{Sp}(1)$-factor into the maximal torus U, this map descends to a map Φ from the homogeneous space W onto Z. Naturally, the map Φ intertwines the left action of the groups through the homomorphism ϕ. We denote the $\mathrm{U}(1)$-left coset of $(t, g, A) \in T^{2n-r} \times G \bowtie \mathrm{Sp}(1)$ by $[t, g, A]$, and the U-left coset of $g \in G$ by $[g]$. We claim that

$$d\Phi_{[1,e,I]} \circ \mathscr{I}_{[1,e,I]} = \mathscr{I}_{[e]} \circ d\Phi_{[1,e,I]}. \tag{10}$$

To verify this claim, let $\mathfrak{h}_0$ be the complexification of the algebra of $\mathrm{U}(1)$. Consider the decomposition $\mathfrak{sp}(1)_{\mathbb{C}} = \mathfrak{h}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$ where $\mathfrak{g}_+$ is the positive root space with respect to $\mathfrak{h}_0$. Then the complexified tangent space of W at $[1, e, I]$ is

$$\begin{aligned} &(2n-r)\,\mathfrak{u}(1)_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_- \\ &= \left((2n-r)\,\mathfrak{u}(1) \oplus \mathfrak{b} \bigoplus_{j=1}^{n} \partial_j \bigoplus_{j=1}^{n} \mathfrak{f}_j\right)_{\mathbb{C}} \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-. \end{aligned} \tag{11}$$

The Abelian part $(2n-r)\,\mathfrak{u}(1)_{\mathbb{C}} \oplus \mathfrak{u}_{\mathbb{C}}$ is the complexification of the algebra $\mathfrak{t}$ of the group E. Since it is in the kernel of $d\Phi$ and it is invariant of I_1, the identity (10) holds on $\mathfrak{t}_{\mathbb{C}}$.

It is observed in [11], Equation (10) and (11) that the (1, 0)-form in $\mathfrak{g}_{\mathbb{C}}$ with respect to I_1 is the direct sum of all positive root spaces, and the (0, 1)-form in $\mathfrak{g}_{\mathbb{C}}$ is the negative root spaces. This decomposition coincides with the type decomposition on the tangent space of Z at the identity coset. Therefore the identity (10) holds on the complement of the Abelian part in $\mathfrak{g}_{\mathbb{C}}$.

Finally, we consider vectors in $\mathfrak{g}_+ \oplus \mathfrak{g}_-$. Since $\Phi([1,e,A]) = [\Delta(A)]$, the restriction of $d\Phi$ to $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ is the map Δ on the Lie algebra. By construction, Δ maps $\mathfrak{g}_+$ to $(2n-r)\,\mathfrak{u}(1)_{\mathbb{C}} \oplus \mathfrak{b}_{\mathbb{C}} \bigoplus_{j=1}^{n} \mathfrak{g}_{\alpha_j}$ and $\mathfrak{g}_-$ to $(2n-r)\,\mathfrak{u}(1)_{\mathbb{C}} \oplus \mathfrak{b}_{\mathbb{C}} \bigoplus_{j=1}^{n} \mathfrak{g}_{-\alpha_j}$. Since $(2n-r)\,\mathfrak{u}(1)_{\mathbb{C}} \oplus \mathfrak{b}_{\mathbb{C}}$ is in the kernel, the quotient of the image of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are contained in $\bigoplus_{j=1}^{n} \mathfrak{g}_{\alpha_j}$ and $\bigoplus_{j=1}^{n} \mathfrak{g}_{-\alpha_j}$ respectively. Since α_j, for $1 \leqq j \leqq n$, is a positive root, the restriction of $d\Phi$ to $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ preserves type decomposition. The identity (10) is verified.

It follows that the map Φ is holomorphic at the identity coset. Since it intertwines the left multiplications, and the complex structures on W and Z are both homogeneous, the map Φ is holomorphic.

Given the definition of Φ through (9), it is not hard to see that the fiber of Φ is the homogeneous space F. To prove that the map Φ is locally trivial, we recall that the space of positive roots $\bigoplus_{\alpha>0} \mathfrak{g}_\alpha$ with the exponential map forms a holomorphic coordinate chart for the identity coset on the flag manifold $Z = G/U$. Any point in the subgroup $E \bowtie \mathrm{Sp}(1)$ is given by the product of an element (t,u,I) in $T^{2n-r} \times U$ and an element $\gamma(A) = (1, \Delta(A)^{-1}, A)$ in $\gamma(\mathrm{Sp}(1))$. It is $(t, u\Delta(A)^{-1}, A)$. We denote its $\mathrm{U}(1)$-coset by its square bracket. Now we consider the map ϱ from $\bigoplus_{\alpha>0} \mathfrak{g}_\alpha \times F$ to W defined by

$$\varrho(v, [t, u\Delta(A)^{-1}, A]) = [t, \exp(v)\, u\Delta(A)^{-1}, A],$$

where v is an element in $\bigoplus_{\alpha>0} \mathfrak{g}_\alpha$. This map is holomorphic as it is holomorphic on both factors. Since the map Φ sends $\varrho(v, [t, u\Delta(A)^{-1}, A])$ to $[\exp(v)]$, ϱ is a trivialization of the map Φ. Since Φ intertwines with left actions, the map Φ is a locally trivial fibration everywhere. q.e.d.

Next we consider the map p from W onto $\mathrm{Sp}(1)/\mathrm{U}(1) = \mathbb{CP}^1$ defined by $p([t,g,A]) = [A]$. It is clear that this map intertwines the left actions. It follows that the map p is holomorphic. This map is precisely the parameterization of the complex structures of the given hypercomplex structure on $T^{2n-r} \times G$. Let S be the inverse image of the identity coset. We have seen in Lemma 4 that there is a holomorphic map from S onto the flag manifold Z. From the proof of Lemma 6, we find the following.

Lemma 7. *The restriction of the map Φ on S is a locally trivial holomorphic fibration from S onto Z with the complex space E as fiber.*

Lemma 8. *The restriction of p to F is a locally trivial holomorphic fibration with E as fiber.*

Proof. The holomorphic chart of the identity coset on $\mathrm{Sp}(1)/\mathrm{U}(1)$ is $\mathfrak{g}_+$. The trivialization is given by the map from $\mathfrak{g}_+ \times E$ to F defined by $(v, [t, u, I]) \mapsto [t, u, \exp(v)]$. q.e.d.

3. Computation of cohomology groups

In this section, we compute the cohomology groups which are relevant to the deformation theory.

Lemma 9. *Let $\mathfrak{t} = H^1(E, \mathcal{O}_E)$. Then for $0 \leqq j \leqq n$, $H^j(E, \mathcal{O}_E) \cong \wedge^j \mathfrak{t}$.*

Lemma 10. *For all j, $R^j p_* \mathcal{O}_F = \mathcal{O}_{\mathbb{CP}^1} \otimes H^j(E, \mathcal{O}_E)$, $R^j \Phi_* \mathcal{O}_S = \mathcal{O}_Z \otimes H^j(E, \mathcal{O}_E)$, and $R^j \Phi_* \mathcal{O}_W = \mathcal{O}_Z \otimes H^j(F, \mathcal{O}_F)$.*

Proof. Since the maps $p_{|F}$, $\Phi_{|S}$ and Φ are locally trivial fibrations, this lemma is a consequence of the Künneth formula and the $\bar{\partial}$-Poincaré lemma. q.e.d.

Lemma 11. *For all j,*

(12) $$H^j(F, \mathcal{O}_F) = H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}) \otimes H^j(E, \mathcal{O}_E) = H^j(E, \mathcal{O}_E),$$

(13) $$H^j(S, \mathcal{O}_S) = H^0(Z, \mathcal{O}_Z) \otimes H^j(E, \mathcal{O}_E) = H^j(E, \mathcal{O}_E),$$

(14) $$H^j(W, \mathcal{O}_W) = H^0(Z, \mathcal{O}_Z) \otimes H^j(F, \mathcal{O}_F) = H^j(E, \mathcal{O}_E).$$

Proof. Consider the Leray spectral sequence with

$$E_2^{p,q} = H^p(\mathbb{CP}^1, R^q p_* \mathcal{O}_F), \quad \text{and} \quad E_\infty^{p,q} \Rightarrow H^{p+q}(F, \mathcal{O}_F).$$

By the last lemma, $E_2^{p,q} = H^p(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}) \otimes H^q(E, \mathcal{O}_E)$. Since the space $H^p(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1})$ vanishes for all $p \geqq 1$, the Leray spectral sequence degenerates at the E_2-terms. The conclusion in (12) follows from the convergence of the spectral sequence. By [2], Proposition 14.10, $H^p(Z, \mathcal{O}_Z)$ vanishes for $p \geqq 1$. Therefore, the above argument is applied to the spectral sequence for $R^p \Phi_* \mathcal{O}_S$ and $R^p \Phi_* \mathcal{O}_W$ to verify (13) and the first equality in (14). The last equality in (14) follows from (12). q.e.d.

Lemma 12. *For all j, $H^j(W, \Phi^* \Theta_Z) = H^0(Z, \Theta_Z) \otimes H^j(E, \mathcal{O}_E)$.*

Proof. The projection formula and Lemma 10 together imply that

$$R^j \Phi_* (\Phi^* \Theta_Z) \cong \Theta_Z \otimes H^j(F, \mathcal{O}_F).$$

By [3], Theorem VII, $H^p(Z, \Theta_Z)$ vanishes for all $p \geqq 1$. Therefore, the spectral sequence of $R^q \Phi_* (\Phi^* \Theta_Z)$ degenerates at the E_2-terms. The convergence of the spectral sequence yields $H^j(W, \Phi^* \Theta_Z) = H^0(Z, \Theta_Z) \otimes H^j(F, \mathcal{O}_F)$. Then the lemma follows from (12). q.e.d.

Lemma 13. *For all j, $H^j(W, p^* \Theta_{\mathbb{CP}^1}) = \mathfrak{sl}(2, \mathbb{C}) \otimes H^j(E, \mathcal{O}_E)$.*

Proof. The tangent sheaf on $\mathbb{CP}^1$ is isomorphic to $\mathcal{O}(2)$. When S_1 and S_2 are two distinct effective divisors for the bundle $p^*\mathcal{O}(1)$, we have the exact sequence

$$0 \to \mathcal{O}_W \to p^*\Theta_{\mathbb{CP}^1} \to \mathcal{O}_{S_1 \cup S_2} \to 0. \tag{15}$$

By Lemma 9 and Lemma 11, the coboundary map from the j-th cohomology level is identical to $\wedge^j \mathfrak{t} \oplus \wedge^j \mathfrak{t} \to \wedge^{j+1}\mathfrak{t}$. Since the bundles involved are homogeneous with respect to the group $T^{2n-r} \times G \bowtie \mathrm{Sp}(1)$, the cohomology spaces are representations of this group, and the coboundary maps are equivariant. Since $\wedge^j \mathfrak{t}$ and $\wedge^{j+1}\mathfrak{t}$ have different weights, by Schur's Lemma the coboundary map is the zero map. Therefore, the induced long exact sequence of (15) splits. We have

$$H^j(W, p^*\Theta_{\mathbb{CP}^1}) = H^j(W, \mathcal{O}_W) \oplus H^j(S_1, \mathcal{O}_{S_1}) \oplus H^j(S_2, \mathcal{O}_{S_2}) = \mathbb{C}^3 \otimes \wedge^j \mathfrak{t}.$$

As the $\mathrm{Sp}(1)$ factor in the semi-direct product acts on the sphere $\mathrm{Sp}(1)/\mathrm{U}(1)$ by the left action, the spaces $H^j(W, \mathcal{O}_W)$, $H^j(S_1, \mathcal{O}_{S_1})$ and $H^j(S_2, \mathcal{O}_{S_2})$ are not acted on invariantly. Therefore, the $\mathbb{C}^3$-factor is an irreducible representation of $\mathrm{Sp}(1)$. It can only be the adjoint representation. q.e.d.

Proposition 3. *Let $\mathcal{D}$ be the kernel sheaf of the differential of the map p from W onto $\mathbb{CP}^1$. For all j,*

$$H^j(W, \Theta_W) = \oplus^n H^j(W, \mathcal{O}_W) \oplus H^0(Z, \Theta_Z) \otimes H^j(E, \mathcal{O}_E) \oplus \mathfrak{sl}(2, \mathbb{C}) \otimes H^j(E, \mathcal{O}_E),$$

$$H^j(W, \mathcal{D}) = \oplus^n H^j(W, \mathcal{O}_W) \oplus H^0(Z, \Theta_Z) \otimes H^j(E, \mathcal{O}_E).$$

Proof. The sheaf $\mathcal{D}$ is defined by the exact sequence

$$0 \to \mathcal{D} \to \Theta_W \xrightarrow{dp} p^*\Theta_{\mathbb{CP}^1} \to 0.$$

We claim that the induced map of dp on the j-th cohomology level, is surjective; and $\ker dp_j = \oplus^n H^j(W, \mathcal{O}_W) \oplus H^j(W, \Phi^*\Theta_Z)$.

To verify these claims, we note that the group $E = T^{2n-r} \times U$ generates n holomorphic vector fields without zeroes on the twistor space W. The complex orbits of the holomorphic action of $T^{2n-r} \times U$ are the transversal intersection of the fibers of the maps Φ and p, we have the exact sequence

$$0 \to \oplus^n \mathcal{O}_W \to \Theta_W \xrightarrow{d\Phi \oplus dp} \Phi^*\Theta_Z \oplus p^*\Theta_{\mathbb{CP}^1} \to 0. \tag{16}$$

By Lemma 11, 12 and 13, the coboundary map from the j-th level is equivalent to

$$H^0(Z, \Theta_Z) \otimes H^j(E, \mathcal{O}_E) \oplus \mathfrak{sl}(2, \mathbb{C}) \otimes H^j(E, \mathcal{O}_E) \to \oplus^n H^{j+1}(E, \mathcal{O}_E).$$

Due to Lemma 9 and [3], Theorem VII, it is equivalent to

$$(\mathfrak{g}_{\mathbb{C}} \otimes \wedge^j \mathfrak{t}) \oplus \big(\mathfrak{sl}(2, \mathbb{C}) \otimes \wedge^j \mathfrak{t}\big) \to \oplus^n \wedge^{j+1}\mathfrak{t} = \mathfrak{t}^* \otimes \wedge^{j+1}\mathfrak{t}. \tag{17}$$

As $\mathfrak{g}$ is semi-simple, $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{sl}(2,\mathbb{C})$ are irreducible representations of $T^{2n-r}\times G \bowtie \mathrm{Sp}(1)$ while $\wedge^{j+1}\mathfrak{t}$ is completely reducible to a direct sum of one-dimensional representations, the coboundary map is the zero map. It follows that the induced long exact sequence of (16) splits, and the induced map of $d\Phi \oplus dp$ on each cohomology level is surjective.

Now the lemma follows from Lemma 12 and Lemma 13. q.e.d.

4. The Kuranishi spaces

Since the second cohomology of the twistor space W does not vanish, we apply Kuranishi theory to calculate deformations [13]. Let us review this theory briefly. As W is a homogeneous manifold with respect to a compact Lie group, it is endowed with a biinvariant metric. Let $\{\beta_1,\dots,\beta_n\}$ be an orthonormal basis of the harmonic representatives of $H^1(W,\Theta_W)$. For any vector $\boldsymbol{t}=(t_1,\dots,t_n)$ in $\mathbb{C}^n$, let $\phi_1(\boldsymbol{t})=t_1\beta_1+\dots+t_n\beta_n$. Let $\mathscr{G}$ be the Green's operator and $\bar\partial^*$ be the adjoint operator of the $\bar\partial$-operator on W. For $v\geqq 2$, define inductively

$$(18)\qquad \phi_v(\boldsymbol{t})=\frac{1}{2}\sum_{\mu=1}^{v-1}\bar\partial^*\mathscr{G}[\phi_\mu(\boldsymbol{t}),\phi_{v-\mu}(\boldsymbol{t})],$$

where the operator $[\cdot,\cdot]$ takes exterior product on the form-components and Lie bracket on the vector-components. Consider the formal sum $\phi(\boldsymbol{t})=\sum_{v\geqq 1}\phi_v$. Let $\{\gamma_1,\dots,\gamma_M\}$ be an orthonormal basis for the space of harmonic $(0,2)$-forms with values in Θ_W. Define

$$(19)\qquad f_k(\boldsymbol{t})=\big([\phi(\boldsymbol{t}),\phi(\boldsymbol{t})],\gamma_k\big).$$

According to the Kuranishi theory, there exists ε such that

$$\{\boldsymbol{t}\in\mathbb{C}^n:|\boldsymbol{t}|<\varepsilon, f_1(\boldsymbol{t})=0,\dots,f_M(\boldsymbol{t})=0\}$$

forms a locally complete family of deformations of W [13], Theorem 1.

Let us consider equivalence of (virtual) deformations before we compute obstructions. By Proposition 3, $H^1(W,\Theta_W)=H^1(E,\mathcal{O}_E)\otimes\big(\mathbb{C}^n\oplus H^0(Z,\Theta_Z)\oplus\mathfrak{sl}(2,\mathbb{C})\big)$. Let us identify each space in this formula. The summand $\mathbb{C}^n$ is generated by the holomorphic action of the group $T^{2n-r}\times B$. It is naturally isomorphic to $\mathfrak{a}_{\mathbb{C}}=H^0(E,\Theta_E)$. On the Abelian variety E, the space $H^1(E,\mathcal{O}_E)$ is isomorphic to the dual $\mathfrak{a}^*_{\mathbb{C}}$ of $H^0(E,\Theta_E)$. By Bott [3], $H^0(Z,\Theta_Z)$ is isomorphic to $\mathfrak{g}_{\mathbb{C}}$. Therefore, Proposition 3 yields

$$(20)\quad H^1(W,\Theta_W)\cong\mathfrak{a}^*_{\mathbb{C}}\otimes\big(\mathfrak{a}\oplus\mathfrak{g}\oplus\mathfrak{sp}(1)\big)_{\mathbb{C}},\ \text{and}\ H^1(W,\mathscr{D})\cong\mathfrak{a}^*_{\mathbb{C}}\otimes(\mathfrak{a}\oplus\mathfrak{g})_{\mathbb{C}}.$$

In particular, when $\mathfrak{a}^*$ is spanned by $\omega_1,\dots,\omega_n$, then the vector space $H^1(W,\Theta_W)$ is complex linearly spanned by $\omega_i\otimes(a_i+v_i+A_i)$ where $a_i\in\mathfrak{a}$, $v_i\in\mathfrak{g}$ and $A_i\in\mathfrak{sp}(1)$.

By Horikawa's deformation theory of maps [9] and the twistor correspondence [14], deformations of hypercomplex structures are contained in $H^1(W,\mathscr{D})$. By comparing this cohomology space with $H^1(W, \Theta_W)$ in Proposition 3, we conclude that the hypercomplex structure on $T^{2n-r} \times G$ deforms in the sub-family defined by $A_i = 0$.

Given $(\boldsymbol{a}, \boldsymbol{v}) = (a_1, \ldots, a_n; v_1, \ldots, v_n)$ in the virtual parameter space of hypercomplex structures, the elements $\{v_1, \ldots, v_n\}$ are contained in the Lie algebra $\mathfrak{v}$ of a maximal torus V of the group G. By Cathelineau [6], equivariant deformations are given by invariant elements of the cohomology spaces. The left-action of the group G on itself generates the adjoint action on $\mathfrak{g}_{\mathbb{C}} \cong H^0(Z, \Theta_Z)$. Via the isomorphism (20), $(\boldsymbol{a}, \boldsymbol{v})$ is a V-invariant element in $H^1(W, \Theta_W)$. Therefore, the deformation generated by $(\boldsymbol{a}, \boldsymbol{v})$ is V-equivariant. Similarly, the deformation generated by $(\boldsymbol{a}, \mathrm{Ad}_g \boldsymbol{v})$ is gVg^{-1}-invariant. As G is compact, maximal tori are conjugate to each other and fixed by the Weyl groups. Therefore, for the purpose of identifying moduli or number of parameters up to equivalence, it suffices to consider the Kuranishi construction for U-invariant elements in the virtual parameter space.

The U-invariant part of $H^1(W, \mathscr{D})$ is the linear space $(\mathfrak{a} \otimes \mathfrak{a}^* \oplus \mathfrak{u} \otimes \mathfrak{a}^*)_{\mathbb{C}}$. Since $\mathfrak{a} \oplus \mathfrak{u}$ is Abelian, when one applies the operator $[\cdot, \cdot]$ to any pair of elements in $(\mathfrak{a} \otimes \mathfrak{a}^* \oplus \mathfrak{u} \otimes \mathfrak{a}^*)_{\mathbb{C}}$, it is identically zero. It follows that the restriction of f_k to the U-invariant part of $H^1(W, \mathscr{D})$ is identically zero. In other words, the U-invariant part of the Kuranishi space is the linear space $(\mathfrak{a} \otimes \mathfrak{a}^* \oplus \mathfrak{u} \otimes \mathfrak{a}^*)_{\mathbb{C}}$.

A deformation of the twistor space W is a twistor space only if the deformation carries a *real structure* [7], Section 2.5, [15], Remark 3.2. If $\phi(\boldsymbol{t}) = \phi_1(\boldsymbol{t})$ in $H^1(W, \Theta_W)$ is in the real linear span of elements of the form $\omega_i \otimes (a_i + v_i + A_i)$, then the induced deformation is invariant of the real structure. Conversely, the infinitesimal deformation of any deformations of twistor spaces is real. Therefore, the Kuranishi family of U-equivariant deformations of the pair (W, p) with real structure is the real vector space $\mathfrak{a}^* \otimes (\mathfrak{a} \oplus \mathfrak{u})$. Due to the correspondence between twistor spaces and hypercomplex structures [15], [17], these deformations correspond to deformations of the underlying hypercomplex structures on $T^{2n-r} \times G$.

Note that deformations in this family have large symmetry groups. Recall that $\mathfrak{a}$ is the algebra of right translations generated by the group $T^{2n-r} \times B$. Since the induced action of $T^{2n-r} \times B$ on $(\mathfrak{a} \oplus \mathfrak{u}) \otimes \mathfrak{a}^*$ is trivial, every element in the complete family is $(T^{2n-r} \times B \times U)$-invariant.

Since $H^0(W, \mathscr{D}) = (\mathfrak{a} \oplus \mathfrak{g})_{\mathbb{C}}$, the real subspace of $(T^{2n-r} \times B \times U)$-invariant elements are precisely the algebra $(2n - r)\,\mathfrak{u}(1) \oplus \mathfrak{b} \oplus \mathfrak{u} = \mathfrak{a} \oplus \mathfrak{u}$. Therefore, the invariant part of $H^0(W, \mathscr{D})$ does not jump throughout an $(T^{2n-r} \times B \times U)$-equivariant deformation. It follows that a local moduli at a generic deformation is a smooth manifold whose dimension is equal to the dimension of the linear space $(\mathfrak{a} \oplus \mathfrak{u}) \otimes \mathfrak{a}^*$. To summarize, we have

Theorem 1. *Suppose G is a compact semi-simple Lie group of rank r, with a maximal torus U and Weyl group $\mathscr{W}$. Let $\mathfrak{g} = \mathfrak{b} \bigoplus_{j=1}^{n} \mathfrak{d}_j \bigoplus_{j=1}^{n} \mathfrak{f}_j$ be the Joyce decomposition of G, and B be the subgroup of G whose Lie algebra is $\mathfrak{b}$. Let $\mathfrak{a}$ be the algebra of right invariant vector fields generated by $T^{2n-r} \times B$. Then $(\mathfrak{a} \otimes \mathfrak{a}^* \oplus \mathfrak{u} \otimes \mathfrak{a}^*)/\mathscr{W}$ is a complete family of deformations of left-invariant hypercomplex structures on $T^{2n-r} \times G$.*

The identity component of the group of hypercomplex symmetries of a generic deformation in this family is the Abelian group T^{2n}.

The local moduli at a generic deformation is a smooth manifold of dimension $n(n+r)$.

The "completeness" here is in the sense of Kuranishi.

The complete family of $T^{2n-r}\times G$-invariant hypercomplex structures on the manifold $T^{2n-r}\times G$ is parametrized by $\mathfrak{a}\otimes\mathfrak{a}^*$. As observed in [10], these n^2 parameters correspond to the choice of bases in the identification (2).

5. Deformations of homogeneous hypercomplex structures

Much of the work in the past three sections can be used to construct deformations of homogeneous hypercomplex structures. In this section, we outline the necessary key points to work through the theory. We begin with Joyce's construction of homogeneous hypercomplex structures with respect to compact simple Lie groups.

Let B_k be the subgroup of G such that its algebra is $\mathfrak{b}_k$ in the Joyce Decomposition of $\mathfrak{g}$. Let G_k be the maximal semi-simple subgroup of B_k. Therefore, the choice of an ordering of roots determines a chain of subgroups.

$$G_n := \{e\} \subset B_n \subset \cdots \subset G_k \subset B_k \subset \cdots \subset G_1 \subset B_1 \subset G_0 := G\,. \tag{21}$$

Note that $\mathfrak{g}_k = \mathfrak{m}_k \bigoplus_{j=k+1}^{n} \mathfrak{d}_j \bigoplus_{j=k+1}^{n} \mathfrak{f}_j$, where $\mathfrak{m}_k$ is a subalgebra of the Abelian algebra $\mathfrak{b}$, and its rank is determined by k. Define $\delta_k = \dim \mathfrak{b}_k - \dim \mathfrak{g}_k$. Apparently,

$$\delta_k \leqq \dim \mathfrak{b} = \operatorname{rank} \mathfrak{g} - n\,.$$

Let P be a subgroup of G such that $G_k \subset P \subset B_k$. Then $\mathfrak{p} = m\mathfrak{u}(1) \oplus \mathfrak{g}_k$, for some $0 \leqq m \leqq \delta_k$. When $\ell = k + m - \delta_k$, the tangent space at the identity coset of the homogeneous space $T^\ell \times G/P$ is

$$\ell\mathfrak{u}(1) \oplus \mathfrak{g}/\mathfrak{p} \cong k\mathfrak{u}(1) \bigoplus_{j=1}^{k} \mathfrak{d}_j \bigoplus_{j=1}^{k} \mathfrak{f}_j \cong \bigoplus_{j=1}^{k} \big(\mathfrak{u}(1) \oplus \mathfrak{d}_j\big) \bigoplus_{j=1}^{k} \mathfrak{f}_j\,.$$

A left-invariant hypercomplex structure at the tangent space of the identity coset is defined as follows. Let $\{e_1, \ldots, e_k\}$ be the standard basis for $\mathbb{R}^n \cong k\mathfrak{u}(1)$. Choose isomorphisms ϕ_j from $\mathfrak{sp}(1)$, the real vector space of imaginary quaternions, to $\mathfrak{d}_j$. It gives a real linear identification from the quaternions $\mathbb{H}$ to $\langle e_j\rangle \oplus \mathfrak{d}_j$. Define the action of I_a on $\mathfrak{f}_j$ by $I_a(v) = [v, \phi_j(\iota_a)]$ where $\iota_1 = i$, $\iota_2 = j$, $\iota_3 = k$. Then a left-invariant hypercomplex structure on $T^\ell \times G/P$ is determined by left translations.

There are two extremes in this construction. The first one is when P is the trivial subgroup of G which we discuss thoroughly in the last three sections. Another is when $P = B_1$. In this case, the hypercomplex manifolds are the compact associated bundles of symmetric quaternionic Kähler manifolds. These hypercomplex manifolds were studied extensively in [11], [15], [16].

To construct the twistor spaces for homogeneous hypercomplex manifolds, we use the homomorphism Δ in Lemma 3 to define a semi-direct product $T^\ell \times G \bowtie \mathrm{Sp}(1)$.

Given the third property in the Joyce Decomposition of $\mathfrak{g}$ and the fact that $\phi_j(\iota_a)$ is in $\mathfrak{d}_j$, we verify that

$$[\Delta(\iota_a), \mathfrak{p}] \in \left[\bigoplus_{j=1}^{n} \mathfrak{d}_j,\ m\mathfrak{u}(1) \oplus \mathfrak{m}_k \bigoplus_{j=k+1}^{n} \mathfrak{d}_j \bigoplus_{j=k+1}^{n} \mathfrak{f}_j \right] \subset \mathfrak{g}_k \subset \mathfrak{p}\,.$$

Therefore, when g is an element of a subgroup P such that $G_k \subset P \subset B_k$, and when A is an element in $\mathrm{Sp}(1)$, then $\Delta(A)g\Delta(A)^{-1}$ is again in P. It follows that $P \bowtie \mathrm{Sp}(1)$ and $P \bowtie \mathrm{U}(1)$ are well-defined subgroups of $T^\ell \times G \bowtie \mathrm{Sp}(1)$.

The smooth manifold underlying the twistor space W for $T^\ell \times G/P$ is the product space $T^\ell \times (G/P) \times S^2$. The two-sphere S^2 is the parameter space of the complex structures in the given hypercomplex structure on $T^\ell \times G/P$. Then the complex structure $\mathscr{I}$ at a point $(t, [g]; v)$ is the direct sum (I_v, J_v) where J_v is the standard complex structure on S^2 defined by the orientation $\{I_1, I_2, I_3\}$. Due to the relation (5) between complex structures, one can verify the following analogue of Lemma 5.

Lemma 14. *The twistor space W of the homogeneous hypercomplex manifold $T^\ell \times G/P$ is the homogeneous complex manifold*

$$\frac{T^\ell \times G \bowtie \mathrm{Sp}(1)}{P \bowtie \mathrm{U}(1)}. \tag{22}$$

The map $p([t, g, A]) = [A]$ from W onto the coset space $\mathrm{Sp}(1)/\mathrm{U}(1)$ is holomorphic. The fibers of this projection are the complex structures of the homogeneous hypercomplex structures on the underlying smooth manifold $T^\ell \times G/P$.

Let $B_k \cdot T^k$ be the subgroup of G such that its complexified algebra is $\mathfrak{b}_k^{\mathbb{C}} \bigoplus_{j=1}^{k} \mathfrak{h}_j^{\mathbb{C}}$. The group $B_k \cdot T^k$ is the commutator of the torus subgroup T^k whose algebra is $\bigoplus_{j=1}^{k} \mathfrak{h}_j$. The quotient space

$$Z = \frac{G}{B_k \cdot T^k} \tag{23}$$

is sometimes called a generalized flag manifold. Then the map ϕ from $T^\ell \times G \bowtie \mathrm{Sp}(1)$ to G defined by $\phi(t, g; A) = g\Delta(A)$ induces an equivariant holomorphic map Φ from the twistor space W to Z.

Since the map Φ intertwines the group actions, and the complex structures are homogeneous, the fibers of the map Φ are holomorphically equivalent. Let us consider the fiber over the identity coset in the generalized flag space. Since

(24) $$\phi^{-1}(B_k \cdot T^k) = \{(t, b\Delta(A)^{-1}; A) : t \in T^\ell,\ b \in B_k \cdot T^k,\ A \in \mathrm{Sp}(1)\},$$

the point $\phi(t, g; A)$ is mapped to $B_k \cdot T^k$ if and only if there exists a point b in $B_k \cdot T^k$ such that $g = b\Delta(A)^{-1}$. On the other hand, the map γ from $\mathrm{Sp}(1)$ to the semi-direct product $T^\ell \times G \bowtie \mathrm{Sp}(1)$ defined by $\gamma(A) = (1, \Delta(A)^{-1}; A)$ is a group homomorphism. The image $\gamma(\mathrm{Sp}(1))$ is contained in the commutator subgroup of $T^\ell \times G$. In particular, the subgroup generated by the product subgroup $B_k \cdot T^k \bowtie \gamma(\mathrm{Sp}(1))$ is a direct product. It is precisely $\Phi^{-1}(B_k \cdot T^k)$. Note that this subgroup of $T^\ell \times G \bowtie \mathrm{Sp}(1)$ contains the group $P \bowtie \mathrm{Sp}(1)$. We shall consider the following homogeneous complex spaces.

(25) $$F = \frac{T^\ell \times B_k \cdot T^k \bowtie \gamma(\mathrm{Sp}(1))}{P \bowtie \mathrm{Sp}(1)}, \quad \text{and} \quad E = \frac{T^\ell \times B_k \cdot T^k}{P}.$$

As seen in Lemma 6 and 7, if S is the inverse image of the identity coset of the projection p from the twistor space onto $\mathbb{CP}^1$, then we have

Lemma 15. *The map $\Phi : W \to Z$ is a locally trivial fibration with fiber F. The restriction of the map Φ on S is a locally trivial holomorphic fibration from S onto Z with the complex space E as fiber.*

The algebraic description of the twistor space W and the relation to the generalized flag manifold are set up so that we can apply the computation and argument in Section 3 and 4 to conclude the following.

Theorem 2. *Suppose G is a compact semi-simple Lie group of rank r, with a maximal torus U and Weyl group $\mathscr{W}$. Let B_k be the subgroup of G with algebra $\mathfrak{b}_k$ from the Joyce Decomposition of the algebra of G, let G_k be the maximal semi-simple subgroup of B_k. Suppose P is a subgroup such that $G_k \subset P \subset B_k$. Let $\mathfrak{a}$ be the algebra of right invariant vector fields generated by $T^\ell \times T^{\delta_k - m}$. Then $(\mathfrak{a} \otimes \mathfrak{a}^* \oplus \mathfrak{u} \otimes \mathfrak{a}^*)/\mathscr{W}$ is a complete family of deformations of the left-invariant hypercomplex structures on $T^\ell \times G/P$.*

The identity component of the group of hypercomplex symmetries of a generic deformation in this family is the Abelian group T^{2k}.

The local moduli at a generic deformation is a smooth manifold of dimension $k(k+r)$.

A special case of this theorem is when P is the subgroup B_1 for a compact simple Lie group. In this case, the hypercomplex structure on $T^1 \times G/B_1$ is the compact associated bundle of the quaternionic Kähler manifold $G/B_1\,\mathrm{Sp}(1)$ [16]. In this way, we improved a result in [15].

Corollary 1. *Suppose that $G/B_1\,\mathrm{Sp}(1)$ is a compact symmetric quaternionic Kähler manifold with positive scalar curvature. Then $(\mathbb{R} \oplus \mathfrak{u})/\mathscr{W}$ is a complete family of deformations of the hypercomplex structure on the compact associated bundle.*

6. Examples

The simplest compact semi-simple Lie groups are products of Sp(1), SO(4) and SO(3). For such manifolds, we construct the global moduli spaces based on the information in Theorem 1, and based on the techniques of constructing Abelian varieties.

6.1. Moduli of hypercomplex structures on $(\mathrm{SU}(2)\times \mathrm{U}(1))^n$. Let

$$(q_1,\dots,q_n;x_1,\dots,x_n)=(\boldsymbol{q};\boldsymbol{x})$$

be coordinates for $\times^n S^3\times\mathbb{R}^n$. Here q_j are unit quaternions. Choose a hypercomplex structure on $\mathbb{H}^n$ by right multiplication of unit quaternions. Then we define a hypercomplex structure on $(S^3\times\mathbb{R})^n$ through the embedding $\times^n S^3\times\mathbb{R}^n\to\times^n S^3\times{}^n\mathbb{R}^+\to\mathbb{H}^n$:

$$(\boldsymbol{q};\boldsymbol{x})\mapsto(q_1,\dots,q_n;e^{x_1},\dots,e^{x_n})\mapsto(e^{x_1}q_1,\dots,e^{x_n}q_n)\,. \tag{26}$$

For $1\leqq j\leqq n$, define an action generated by

$$\gamma_j(\boldsymbol{q};\boldsymbol{x})=(e^{2\pi i\theta_{1j}}q_1,\dots,e^{2\pi i\theta_{nj}}q_n;\boldsymbol{x}+\boldsymbol{v}_j)\,. \tag{27}$$

The action of γ_j is represented by the column vectors $\boldsymbol{v}_j$ and $\Theta_j=(\theta_{1j},\dots,\theta_{nj})^T$, where θ_{ij} are in $\mathbb{R}/\mathbb{Z}$.

Assume that the vectors $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_n\}$ are linearly independent. Let $\Gamma\cong\mathbb{Z}^n$ be the group generated by $\{\gamma_1,\dots,\gamma_n\}$. We call

$$(\Theta\,|\,V)=(\Theta_1,\dots,\Theta_n\,|\,\boldsymbol{v}_1,\dots,\boldsymbol{v}_n) \tag{28}$$

the *period matrix* of the manifold $(S^3\times\mathbb{R})^n/\Gamma$. The groups Γ are parameterized by the *space of rank-n period matrices*

$$\mathscr{P}^n=(\mathbb{R}/\mathbb{Z})^{n^2}\times \mathrm{GL}(n,\mathbb{R})\,. \tag{29}$$

However, different period matrices may generate the same group. In fact, Γ is generated by $\{\gamma_1,\dots,\gamma_n\}$ and $\{\hat\gamma_1,\dots,\hat\gamma_n\}$ if and only if there is a matrix $M=(m_{ij})$ in $\mathrm{GL}(n,\mathbb{Z})$ such that $\hat\gamma_k=\gamma_1^{m_{1k}}\cdots\gamma_n^{m_{nk}}$. It follows that the period matrices $(\Theta\,|\,V)$ and $(\hat\Theta\,|\,\hat V)$ generate the same group if and only if

$$(\hat\Theta\,|\,\hat V)=(\Theta M\,/\,VM)\,. \tag{30}$$

The quotient space $(S^3\times\mathbb{R})^n/\Gamma$ is a hypercomplex manifold because the actions of Γ commute with the right multiplications of the quaternions on $(q_1,\dots,q_n)$. The quotient space is diffeomorphic to $(S^3\times S^1)^n$ through the map from $(S^3\times\mathbb{R})^n$ to $(S^3\times S^1)^n$: when $\boldsymbol{y}=V^{-1}\boldsymbol{x}$,

$$(\boldsymbol{q};\boldsymbol{x})\mapsto(e^{-2\pi i(\sum_j y_j\theta_{j1})}q_1,\dots,e^{-2\pi i(\sum_j y_j\theta_{jn})}q_n;e^{2\pi i y_1},\dots,e^{2\pi i y_n})\,. \tag{31}$$

Lemma 16. *The hypercomplex manifolds* $(S^3 \times \mathbb{R})^n/\Gamma$ *and* $(S^3 \times \mathbb{R})^n/\Gamma'$ *are equivalent if and only if there exist period matrices* $(\Theta\,|\,V)$ *and* $(\Theta'\,|\,V')$ *for* Γ *and* Γ' *respectively such that* $V = V'$, *and* $\Theta_j = \pm\,\Theta'_j$.

Proof. Let W be the twistor space of $(S^3 \times \mathbb{R})^n$. The actions of Γ and Γ' are uniquely lifted to holomorphic actions on W. The twistor spaces for $(S^3 \times \mathbb{R})^n/\Gamma$ and $(S^3 \times \mathbb{R})^n/\Gamma'$ are W/Γ and W/Γ' respectively. Assume that there is a hyper-holomorphic isomorphism from $(S^3 \times \mathbb{R})^n/\Gamma$ to $(S^3 \times \mathbb{R})^n/\Gamma'$; it lifts to a holomorphic isomorphism between the twistor spaces W/Γ and W/Γ'. Such an isomorphism lifts to a holomorphic automorphism on W intertwining the groups Γ and Γ'.

Through the map $\times^n S^3 \times \mathbb{R}^n \to \times^n S^3 \times {}^n\mathbb{R}^+ \to \mathbb{H}^n \subset \mathbb{HP}^n$ defined by

$$(\boldsymbol{q}; \boldsymbol{x}) \mapsto (q_1, \ldots, q_n; e^{x_1}, \ldots, e^{x_n}) \mapsto [1, e^{x_1} q_1, \ldots, e^{x_n} q_n], \tag{32}$$

the hypercomplex manifold $(S^3 \times \mathbb{R})^n$ is a quaternionic submanifold of $\mathbb{H}^n \subset \mathbb{HP}^n$. It follows that its twistor space is a complex analytic subspace of the twistor space $\mathbb{CP}^{2n+1}$ of $\mathbb{HP}^n$. Since the co-dimension of W in $\mathbb{CP}^{2n+1}$ is equal to two, by the Hartog Theorem, the analytic automorphism on W is uniquely extended to an analytic automorphism of $\mathbb{CP}^{2n+1}$. In particular, it is a linear map. It descends to a linear quaternionic map f from $\mathbb{HP}^n$ to $\mathbb{HP}^n$. Since the map f sends $(S^3 \times \mathbb{R})^n$ to $(S^3 \times \mathbb{R})^n$, it sends quaternionic coordinate hyperplanes to quaternionic coordinate hyperplanes. Therefore, f is in

$$\big(\mathrm{GL}(1, \mathbb{H})\big)^{n+1} = \mathrm{GL}(1, \mathbb{H}) \times \cdots \times \mathrm{GL}(1, \mathbb{H}).$$

Since f is a hypercomplex transformation from $\mathbb{H}^n$ to $\mathbb{H}^n$, it is contained in $\{1\} \times \big(\mathrm{GL}(1, \mathbb{H})\big)^n$. Therefore, there exists $(r_i; q_i)$ in $(\mathbb{R}^+)^n \times \big(\mathrm{Sp}(1)\big)^n \cong \big(\mathrm{GL}(1, \mathbb{H})\big)^n$ such that $f(p_1, \ldots, p_n; x_1, \ldots, x_n) = (q_1 p_1, \ldots, q_n p_n; \ln r_1 + x_1, \ldots, \ln r_n + x_n)$. This map intertwines the group Γ and Γ' if and only if there exist generators $\{\gamma_1, \ldots, \gamma_n\}$ and $\{\gamma'_1, \ldots, \gamma'_n\}$ for these two groups respectively such that $\gamma_j \circ f = f \circ \gamma'_j$. It follows that for each j, $\boldsymbol{v}_j = \boldsymbol{v}'_j$ and $q_\ell e^{2\pi i \theta_{\ell j}} = e^{2\pi i \theta'_{\ell j}} q_\ell$ for each ℓ. It is possible if and only if q_ℓ is the unit quaternion j and $e^{2\pi i \theta_{\ell j}}$ is the complex conjugate of $e^{2\pi i \theta'_{\ell j}}$. Therefore, the lemma follows. q.e.d.

Combining the last lemma with the identification given in (30), we conclude that the quotient space $\mathscr{M} = \mathscr{P}^n/\big(\mathbb{Z}_2^n \times \mathrm{GL}(n, \mathbb{Z})\big)$ is a moduli space for hypercomplex structures on the product manifold $(S^3 \times S^1)^n$. In view of the dimension count in Theorem 1, we conclude that this moduli space is complete.

Theorem 3. *The quotient space* $\mathscr{M} = \mathscr{P}^n/\big(\mathbb{Z}_2^n \times \mathrm{GL}(n, \mathbb{Z})\big)$ *is a complete moduli space for hypercomplex structures on the product manifold* $(S^3 \times S^1)^n$.

The set of singular points of the moduli space contains the quotients of the period matrices with non-trivial isotropy with respect to the action of $\mathbb{Z}_2^n$. The most singular part is given by the fixed points of the action of $\mathbb{Z}_2^n$. This is the space of left-invariant hypercomplex structures. The stratification of the singularities of the moduli space corresponds to the size of symmetries. For instance, a period matrix is a fixed point of the j-th $\mathbb{Z}_2$-action if and only if the corresponding hypercomplex structure allows the left multiplication of the j-th factor in $\big(\mathrm{SU}(2) \times \mathrm{U}(1)\big)^n$ to be a group of hypercomplex transformations.

The complete moduli space of hypercomplex structures on $S^3 \times S^1$ is the quotient $\{re^{2\pi i\theta} : 0 < r < 1\}/\mathbb{Z}_2$ defined by the equivalent relation $re^{2\pi i\theta} = re^{-2\pi i\theta}$. It is $\{re^{2\pi i\theta} : 0 < r < 1, 0 \leqq \theta \leqq 0.5\}$. This result refines our observations in 1.1.

6.2. Products of SO(4) and SO(3). The results in the last section can be used to find the complete moduli space of hypercomplex structures on $(T^2 \times \mathrm{SO}(4))^m$ because $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is a double covering of SO(4). Let a $\mathbb{Z}_2^m$ action on $\times^{2m} S^3 \times \mathbb{R}^{2m}$ be generated by the maps $(q_1, \dots, -q_{2i-1}, -q_{2i}, \dots, q_{2m}; x_1, \dots, x_{2m})$ for $1 \leqq i \leqq m$. Then the quotient space $(S^3 \times \mathbb{R})^{2m}/(\Gamma \times \mathbb{Z}_2^m)$ is the manifold $(T^2 \times \mathrm{SO}(4))^m$ with a hypercomplex structure. Let $\mathbb{Z}_2^m$ act on the space of period matrices $\mathscr{P}^{2m}$ by

$$(\Theta_1, \dots, \Theta_{2i-1} + 1/2, \Theta_{2i} + 1/2, \dots, \Theta_{2m}; \mathfrak{v}_1, \dots, \mathfrak{v}_{2m}).$$

Then $\mathscr{P}^{2m}/(\mathbb{Z}_2^{2m} \times \mathrm{GL}(2m, \mathbb{R}) \times \mathbb{Z}_2^m)$ is a complete moduli space of hypercomplex structures on $(T^2 \times \mathrm{SO}(4))^m$.

The complete moduli spaces of hypercomplex structures on

$$(\mathrm{SU}(2) \times S^1)^n \times (T^2 \times \mathrm{SO}(4))^m \times (\mathrm{SO}(3) \times S^1)^\ell$$

can be similarly constructed.

6.3. SU(3) as a hypercomplex quotient of $S^1 \times S^{11}$. In our previous paper [15], we studied the deformations of the hypercomplex structures on SU(3) from a different perspective. From Theorem 1, $\mathfrak{u}(1) \oplus (\mathfrak{u}(1) \oplus \mathfrak{u}(1))/\mathscr{W}$ is a complete parameter space of hypercomplex structures on SU(3).

These parameters can be realized from the hypercomplex quotients as follows. We choose a hypercomplex structure on $\mathbb{C}^3 \oplus \mathbb{C}^3$ by $I_1(\chi, \ell) = (i\chi, -i\ell)$, $I_2(\chi, \ell) = (i\ell, i\chi)$, $I_3(\chi, \ell) = (-\ell, \chi)$. For any real number r in the open interval (0, 1), let $\langle r \rangle$ be the integer group generated by $(r\chi, r\ell)$. It is a group of hypercomplex automorphisms. Through the map

$$(\chi, \ell) \mapsto \left(e^{2\pi i \frac{\ln\sqrt{|\chi|^2 + |\ell|^2}}{\ln r}}; \left(\frac{\sqrt{2}\chi}{\sqrt{|\chi|^2 + |\ell|^2}}, \frac{\sqrt{2}\ell}{\sqrt{|\chi|^2 + |\ell|^2}}\right)\right), \tag{33}$$

the quotient space $H(r)$ of $\mathbb{C}^6 \setminus \{0\}$ with respect to the integer group $\langle r \rangle$ is a hypercomplex manifold diffeomorphic to the product of a unit circle with the 11-dimensional sphere of radius $\sqrt{2}$, $S^1 \times S^{11}(\sqrt{2})$.

For a generic element $\eta = 2\pi i \left(\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}\right)$ near $2\pi i(1, 1, 1)$ in the Cartan subalgebra $\mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ in $\mathfrak{u}(3)$, its period is the product $r_1 r_2 r_3$. Let $\Gamma(\eta)$ be a group acting on $S^1 \times S^{11}(\sqrt{2})$ defined by

$$(e^{2\pi i\varrho}, \chi, \ell) \mapsto \left(e^{2\pi i\left(\frac{t}{r_1 r_2 r_3} + \varrho\right)}, \exp(t\eta)\chi, \exp(t\eta)\ell\right). \tag{34}$$

It is a group of hypercomplex automorphisms of $H(e^{r_1 r_2 r_3})$.

We consider $(e^{2\pi i\varrho}, \chi, \ell)$, with $|\chi|^2 + |\ell|^2 = 2$, as 'coordinates' for $S^1 \times S^{11}(\sqrt{2})$, and define on it a function $\mu = (\mu_1, \mu_2, \mu_3)$ by $\mu_1(\chi, \ell) = |\chi|^2 - |\ell|^2$ and

$$(\mu_2 + i\mu_3)(\chi, \ell) = 2\langle \chi, \ell \rangle,$$

where $\langle \chi, \ell \rangle$ is the Hermitian inner product on $\mathbb{C}^3$. The function μ satisfies the "Cauchy-Riemann condition" for it being a hypercomplex moment map [10].

Since the vector field generated by $\Gamma(\eta)$ satisfies the "transversality condition" for it being a hypercomplex moment map [10], the quotient space

$$M(\eta) = \mu^{-1}(0)/\Gamma(\eta)$$

is a hypercomplex manifold.

To verify that $M(\eta)$ is diffeomorphic to $\mathrm{SU}(3)$, note that the invariant functions of the action of $\Gamma(\eta)$ are in the pair of vectors

$$(e^{-2\pi i r_2 r_3 \varrho}\chi_1, e^{-2\pi i r_3 r_1 \varrho}\chi_2, e^{-2\pi i r_1 r_2 \varrho}\chi_3)^T, \quad \text{and} \quad (e^{-2\pi i r_2 r_3 \varrho}\ell_1, e^{-2\pi i r_3 r_1 \varrho}\ell_2, e^{-2\pi i r_1 r_2 \varrho}\ell_3)^T.$$

Therefore, the quotient space $M(\eta)$ is the space of pairs of vectors (χ, ℓ) in $\mathbb{C}^3$ satisfying the following conditions:

$$(35) \qquad |\chi|^2 + |\ell|^2 = 2, \quad |\chi|^2 - |\ell|^2 = 0, \quad \langle \chi, \ell \rangle = 0.$$

Through the map $(\chi, \ell, \bar{\chi} \times \bar{\ell})$, we obtain an element in $\mathrm{SU}(3)$.

Since the action of $\Gamma(\eta)$ on $\mathrm{SU}(3)$ is from the left, the corresponding infinitesimal deformation coincides with the deformation determined by η when we use the identification of Proposition 3. By Theorem 1, the algebra $3\mathfrak{u}(1)$ is a complete parameter space. When $\eta = 2\pi i(1, 1, 1)$, the hypercomplex structure is invariant of the left action of $\mathrm{SU}(3)$, therefore, any small deformation of such hypercomplex structures on $\mathrm{SU}(3)$ can be obtained as a hypercomplex quotient. However, finding global moduli remains a difficult problem as seen in [5].

This construction should be compared with the one in [15], Example 6.4, where we constructed deformations of the compact associated bundle of $\mathbb{CP}^2$.

6.4. Moduli space of $T^1 \times \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$. In this section, we construct the moduli space of hypercomplex structures on $M = T^1 \times \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$ to illustrate the theory in Section 5. It is also an improvement of results in [15].

Consider $\mathbb{H}^n$ as a quaternionic right module so that the scalar multiplication is from the right. The quaternionic inner product is $\langle \vec{q}, \vec{p} \rangle = \sum_{a=1}^{n} \bar{q}_a p_a$. Then the space $\mathrm{Sp}(n)/\mathrm{Sp}(n-1)$ is the space of unit quaternion vectors.

Let $D(a_1, \ldots, a_n)$ be the $n \times n$ diagonal matrix whose entries are as indicated. Choose $\{\theta_1, \ldots, \theta_n\}$ in $\mathbb{R}/\mathbb{Z}$. Choose any non-zero number v in $\mathbb{R}$. Define an action γ on $\mathbb{R} \times \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$ by

(36) $$\gamma(x, \vec{q}) = \big(x + v, D(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n})\,\vec{q}\big).$$

γ generates an integer group $\Gamma = \langle \gamma \rangle$.

To describe the quotient space $X_\Gamma := \big(\mathbb{R} \times \mathrm{Sp}(n)/\mathrm{Sp}(n-1)\big)/\Gamma$, consider the map

(37) $$(x, \vec{q}) \mapsto \big(e^{2\pi i\frac{x}{v}}; D(e^{-2\pi i\theta_1\frac{x}{v}}, \ldots, e^{-2\pi i\theta_n\frac{x}{v}})\,\vec{q}\big).$$

It factors through the quotient of $\mathbb{R} \times \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$. It follows that the quotient space X_Γ is diffeomorphic to $M = S^1 \times \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$.

There exists a hypercomplex structure on the quotient manifold X_Γ for the following reasons. The space $\mathbb{R} \times \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$ is embedded in $\mathbb{H}^n \backslash \{0\}$ via the map $(x, \vec{q}) \mapsto e^x\vec{q}$. The right multiplication of quaternions on $\mathbb{H}^n$ defines a hypercomplex structure on it. Since the embedding is invariant by right multiplication, $\mathbb{R} \times \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$ inherits a hypercomplex structure which can also be seen as right multiplication of quaternions on the column vector $\vec{q}$. As the action of Γ is from the left, it commutes with the quaternionic multiplication. Therefore, the hypercomplex structure on $\mathbb{R} \times \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$ descends to a hypercomplex structure on X_Γ.

A hypercomplex isomorphism from X_Γ to $X_{\hat{\Gamma}}$ is lifted to a hypercomplex automorphism f on $\mathbb{H}^n \backslash \{0\}$ intertwining the action of Γ and $\hat{\Gamma}$. By embedding $\mathbb{H}^n \backslash \{0\}$ into $\mathbb{HP}^n$, we find that the space of hypercomplex automorphisms on $\mathbb{H}^n \backslash \{0\}$ is $\mathrm{GL}(n, \mathbb{H})$. Therefore, f is an $n \times n$-matrix of quaternions (f_{ab}). Since f intertwines the groups Γ and $\hat{\Gamma}$, there exist generators $\gamma = \big(v|(\theta_1, \ldots, \theta_n)^T\big)$ and $\hat{\gamma} = \big(\hat{v}|(\hat{\theta}_1, \ldots, \hat{\theta}_n)^T\big)$ such that $\gamma\big(f(x, \vec{q})\big) = f\big(\hat{\gamma}(x, \vec{q})\big)$. This is equivalent to

(38) $$e^{2\pi i\theta_a} f_{ab}\, e^v = f_{ab}\, e^{2\pi i\hat{\theta}_b} e^{\hat{v}}$$

for all $1 \leqq a, b \leqq n$. Since the matrix f is non-singular, by calculating the norms, we deduce that $v = \hat{v}$. The constraints (38) are equivalent to $Df = f\hat{D}$. Let $\vec{f_b}$ be the b-th column of the matrix f. The constraints (38) are equivalent to $D\vec{f_b} = \vec{f_b}\, e^{2\pi i\hat{\theta}_b}$ for all $1 \leqq b \leqq n$. Without loss of generality, we assume that the length of $\vec{f_b}$ is equal to 1. Then Equation (38) is solved when f_{ab} is equal to $0, 1$ or the unit quaternion j. If $f_{ab} = 1$, then $\theta_a = \hat{\theta}_b$. If $f_{ab} = j$, then $\theta_a = -\hat{\theta}_b$. As $|\vec{f_b}| = 1$, either $\vec{f_b} = e_a$ or je_a for some $1 \leqq a \leqq n$. Since the matrix f is in $\mathrm{GL}(n, \mathbb{H})$, when $c \neq b$, then $\vec{f_c} = e_d$ or je_d for some $d \neq a$. Therefore,

(39) $$f = (j^{\delta_1} e_{\varepsilon(1)}, \ldots, j^{\delta_n} e_{\varepsilon(n)})$$

where ε is an element in the permutation group S_n of n-elements, and

(40) $$(\hat{\theta}_1, \ldots, \hat{\theta}_n) = (\pm\theta_{\varepsilon(1)}, \ldots, \pm\theta_{\varepsilon(n)}).$$

This group of identifications is precisely the semi-direct product $\mathbb{Z}_2^n \bowtie S_n$ with S_n acting on $\mathbb{Z}_2^n$, i.e. the Weyl group of $\mathrm{Sp}(n)$.

Since $\mathrm{GL}(1,\mathbb{Z}) = \{\pm I\}$, the quotient of $\mathbb{R} \times (\mathbb{R}/\mathbb{Z})^n$ with respect to $\mathrm{GL}(1,\mathbb{Z})$ is $\mathbb{R}^+ \times (\mathbb{R}/\mathbb{Z})^n$. In conclusion, we have

Theorem 4. *The moduli space of hypercomplex structures on* $T^1 \times \dfrac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}$ *is the quotient space* $\mathbb{R}^+ \times \dfrac{U}{\mathscr{W}}$ *where* U *is a maximal torus of the group* $\mathrm{Sp}(n)$ *and* $\mathscr{W}$ *is the corresponding Weyl group.*

We believe that the moduli space of hypercomplex structures containing the left-invariant hypercomplex structures on $S^1 \times \dfrac{G}{B_1}$, for any compact simple Lie group G, is the space $\mathbb{R}^+ \times \dfrac{U}{\mathscr{W}}$ where U is a maximal torus and $\mathscr{W}$ is the corresponding Weyl group.

6.5. Remarks about $T^n \times \mathrm{Sp}(n)$. The inclusion of $\times^n \mathrm{Sp}(1)$ in $\mathrm{Sp}(n)$ as a diagonal subgroup gives an identification of the Kuranishi spaces of $(T^n \times T^n)$-equivariant hypercomplex structures of $T^n \times {}^n\mathrm{Sp}(1)$ and $T^n \times \mathrm{Sp}(n)$. Therefore, the local moduli of hypercomplex structures on $T^n \times {}^n\mathrm{Sp}(1)$ and $T^n \times \mathrm{Sp}(n)$ are identical. This relation may be used to construct the global moduli space of hypercomplex structures on $T^n \times \mathrm{Sp}(n)$, much as in Example 5.1.

Such comparison between deformations of hypercomplex structures on $T^{2n-r} \times G$ and deformations of hypercomplex structures on a subgroup occurs in a more general context. If the algebra $\mathfrak{b}$ is trivial in the Joyce Decomposition of G, let K be a subgroup of G such that its Lie algebra is $\mathfrak{d}_n \oplus \cdots \oplus \mathfrak{d}_1$. Then K is a semi-simple Lie group, and $T^{2n-r} \times K$ is a hypercomplex submanifold of $T^{2n-r} \times G$ with respect to the left-invariant hypercomplex structures. The inclusion map from $T^{2n-r} \times K$ to $T^{2n-r} \times G$ induces a one-to-one correspondence between the U-equivariant hypercomplex deformations of $T^{2n-r} \times K$ and of $T^{2n-r} \times G$. Since the group K is a product of SO(4)'s, SO(3)'s and Sp(1)'s, one may again attempt to construct the global moduli space for the manifold $T^{2n-r} \times G$ using Example 5.2.

This idea of using subgroups to find global moduli for the ambient group has its limitation. For example, the subgroup K in $\mathrm{SU}(3)$ with algebra $\mathfrak{b} \oplus \mathfrak{d}_1$ in the Joyce decomposition is $\mathrm{U}(2)$. However, the moduli of hypercomplex structure on $\mathrm{U}(2)$ is two-dimensional while the local moduli for $\mathrm{SU}(3)$ is three-dimensional.

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