Commun. Math. Phys. 213, 19 – 37 (2000)

Communications in
Mathematical Physics

Geometry of Hyper-Kähler Connections with Torsion

Gueo Grantcharov***, Yat Sun Poon****

Department of Mathematics, University of California at Riverside, Riverside, CA 92521, USA.
E-mail: geogran@math.ucr.edu; ypoon@math.ucr.edu

Received: 1 October 1999 / Accepted: 30 January 2000

Abstract: The internal space of a $\mathrm{N} = 4$ supersymmetric model with Wess–Zumino term has a connection with totally skew-symmetric torsion and holonomy in $\mathrm{Sp}(n)$. We study the mathematical background of this type of connection. In particular, we relate it to classical Hermitian geometry, construct homogeneous as well as inhomogeneous examples, characterize it in terms of holomorphic data, develop its potential theory and reduction theory.

1. Introduction

It has been known that the internal space for the $\mathrm{N} = 2$ supersymmetric one-dimensional sigma model is a Kähler manifold [28], and the internal space for the $\mathrm{N} = 4$ supersymmetric one-dimensional sigma model is a hyper-Kähler manifold [6, 15]. It means that there exists a torsion-free connection with holonomy in $\mathrm{U}(n)$ or $\mathrm{Sp}(n)$, respectively, on the internal space.

It has also been known for a fairly long time that when the Wess–Zumino term is present in the sigma model, the internal space has linear connections with holonomy in $\mathrm{U}(n)$ or $\mathrm{Sp}(n)$ depending on the numbers of supersymmetry. However, the connection has torsion and the torsion tensor is totally skew-symmetric [11, 16, 14]. The geometry of a connection with totally skew-symmetric torsion and holonomy in $\mathrm{U}(n)$ is referred to as KT-geometry by physicists. When the holonomy is in $\mathrm{Sp}(n)$, the geometry is referred to as HKT-geometry.

If one ignores the metric and the connection of a HKT-geometry, the remaining object on the manifold is a hypercomplex structure. The subject of hypercomplex manifolds has been studied by many people since the publication of [24] and [4]. A considerable

---

* On leave from University of Sofia. Partially supported by Contract MM 809/1998 with Ministry of Education of Bulgaria and by Contract 238/1998 with the University of Sofia.

** Partially supported by the IHES.

amount of information is known. It has a twistor correspondence [24, 21]. There are homogeneous examples [19] and there are also inhomogeneous examples [5, 22]. There is a reduction construction modeled on symplectic reduction and hyper-Kähler reduction [18]. However, all these works focus on the hypercomplex structure and the associated Obata connection which is a torsion-free connection preserving the hypercomplex structure. What is not discussed in these works is hyper-Hermitian geometry.

On the other hand, Hermitian connections on almost Hermitian manifolds are studied rather thoroughly by Gauduchon [12]. He considered a subset of Hermitian connections determined by the form of their torsion tensor, called canonical connections.

Guided by physicists' work and based on the results on hypercomplex manifolds, we review and further develop the theory of HKT-geometry. Some of our observations are re-interpretation of physicists' results, especially those in [16, 17] and [20, 26], and some of the results in this paper are new.

In Sect. 2, we review the basic definition of HKT-geometry along the line of classical Hermitian geometry developed by Gauduchon [12]. Based on Joyce's construction of homogeneous hypercomplex manifolds [19], we review the construction of homogeneous HKT-geometry with respect to compact semi-simple Lie groups [20].

In Sect. 3, we find that a hyper-Hermitian manifold admits a HKT-connection if and only if for each complex structure, there is a holomorphic (0,2)-form. This characterization easily implies that some hyper-Hermitian structures are not HKT-structure. Furthermore, when this characterization is given a twistorial interpretation, the associated object on the twistor space of the hypercomplex structure is holomorphic with respect to a non-standard almost complex structure $\mathcal{J}_2$. This almost complex structure $\mathcal{J}_2$ is first discussed by Eells and Salamon in a different context [9]. Since this almost complex structure is never integrable, we focus on the holomorphic (0,2)-forms. From this perspective, we verify that there are HKT-structures on nilmanifolds, and that the twist of a HKT-manifold is again a HKT-manifold.

In Sect. 4 we study the potential theory for HKT-geometry which is based on results in Sect. 3. We shall see that local HKT-geometry is very flexible in the sense that the existence of one generates many through a perturbation of potential functions. In particular, we show that hyper-Kähler potentials generate many HKT-potentials. The results in this section and Sect. 3 allow us to construct a large family of inhomogeneous HKT-structures on compact manifolds including $S^1 \times S^{4n+3}$.

Finally, a reduction theory based on hyper-Kähler reduction for HKT-geometry is developed in Sect. 5.

2. Hyper-Kähler Geometry with Torsion

2.1. Kähler Geometry with Torsion. Let M be a smooth manifold with Riemannian metric g and an integrable complex structure J. It is a Hermitian manifold if $g(JX, JY) = g(X, Y)$. The Kähler form F is a type (1,1)-form defined by $F(X, Y) = g(JX, Y)$.

A linear connection ∇ on M is Hermitian if it preserves the metric g and the complex structure J. i.e.,

$$\nabla g = 0 \text{ and } \nabla J = 0.$$

Since the connection preserves the metric, it is uniquely determined by its torsion tensor T. We shall also consider the following (3,0)-tensor:

$$c(X, Y, Z) = g(X, T(Y, Z)). \quad (1)$$

Gauduchon found that on any Hermitian manifold, the collection of canonical Hermitian connections is an affine subspace of the space of linear connections [12]. This affine subspace is at most one dimensional. It is one point if and only if the Hermitian manifold is Kähler, i.e., when the Kähler form is closed, then the family of canonical Hermitian connections collapses to the Levi-Civita connection of the given metric. It is one-dimensional if and only if the Hermitian manifold is non-Kähler. In the latter case, there are several distinguished Hermitian connections. For example, the Chern connection and Lichnerwicz's *first canonical connection* are in this family. We are interested in another connection in this family.

Physicists find that the presence of the Wess–Zumino term in $\mathrm{N} = 2$ supersymmetry yields a Hermitian connection whose torsion c is totally skew-symmetric. In other words, c is a 3-form. Such a connection turns out to be another distinguished Hermitian connection [3,12]. The geometry of such a connection is called by physicists a KT-connection. Among some mathematicians, this connection is called the Bismut connection. According to Gauduchon [12], on any Hermitian manifold, there exists a unique Hermitian connection whose torsion tensor c is a 3-form. Moreover, the torsion form can be expressed in terms of the complex structure and the Kähler form. Recall the following definitions and conventions [2, Eqs. 2.8 and 2.15–2.17]. For any n-form ω, when

$$(J\omega)(X_1,\dots,X_n) := (-1)^n\omega(JX_1,\dots,JX_n) \quad \text{then} \quad d^c\omega = (-1)^n JdJ\omega, \tag{2}$$

and

$$\partial = \frac{1}{2}(d+id^c) = \frac{1}{2}(d+(-1)^n iJdJ), \quad \overline{\partial} = \frac{1}{2}(d-id^c) = \frac{1}{2}(d-(-1)^n iJdJ). \tag{3}$$

By [12], the torsion 3-form of the Bismut connection is

$$c(X,Y,Z) = -\frac{1}{2}d^cF(X,Y,Z). \tag{4}$$

2.2. Hyper-Kähler Connection and HKT-Geometry. Three complex structures I_1, I_2 and I_3 on M form a hypercomplex structure if

$$I_1^2 = I_2^2 = I_3^2 = -1, \quad \text{and} \quad I_1I_2 = I_3 = -I_2I_1. \tag{5}$$

A triple of such complex structures is equivalent to the existence of a 2-sphere worth of integrable complex structures:

$$\mathcal{I} = \{a_1I_1 + a_2I_2 + a_3I_3 : a_1^2 + a_2^2 + a_3^2 = 1\}. \tag{6}$$

When g is a Riemannian metric on the manifold M such that it is Hermitian with respect to every complex structure in the hypercomplex structure, $(M,\mathcal{I},g)$ is called a hyper-Hermitian manifold. Note that g is hyper-Hermitian if and only if

$$g(X,Y) = g(I_1X,I_1Y) = g(I_2X,I_2Y) = g(I_3X,I_3Y). \tag{7}$$

On a hyper-Hermitian manifold, there are two natural torsion-free connections, namely the Levi-Civita connection and the Obata connection. However, in general, the Levi-Civita connection does not preserve the hypercomplex structure and the Obata connection does not preserve the metric. We are interested in the following types of connections:

Definition 1. *A linear connection* ∇ *on a hyper-Hermitian manifold* $(M, \mathcal{I}, g)$ *is hyper-Hermitian if*

$$\nabla g = 0, \quad \textit{and} \quad \nabla I_1 = \nabla I_2 = \nabla I_3 = 0. \tag{8}$$

Definition 2. *A linear connection* ∇ *on a hyper-Hermitian manifold* $(M, \mathcal{I}, g)$ *is hyper-Kähler if it is hyper-Hermitian and its torsion tensor is totally skew-symmetric.*

A hyper-Kähler connection is referred to HKT-connection in the physics literature. The geometry of this connection or this connection is also referred to HKT-geometry.

Note that a HKT-connection is also the Bismut connection for each complex structure in the given hypercomplex structure. For the complex structures $\{I_1, I_2, I_3\}$, we consider their corresponding Kähler forms $\{F_1, F_2, F_3\}$ and the complex operators $\{d_1, d_2, d_3\}$, where $d_i = {d_i}^c$. Due to Gauduchon's characterization of the Bismut connection, we have

Proposition 1. *A hyper-Hermitian manifold* $(M, \mathcal{I}, g)$ *admits a hyper-Kähler connection if and only if* $d_1 F_1 = d_2 F_2 = d_3 F_3$*. If it exists, it is unique.*

In view of the uniqueness, we say that $(M, \mathcal{I}, g)$ is a HKT-structure if it admits a hyper-Kähler connection. If the hyper-Kähler connection is also torsion-free, then the HKT-structure is a hyper-Kähler structure.

2.3. Homogeneous Examples. Due to Joyce [19], there is a family of homogeneous hypercomplex structures associated to any compact semi-simple Lie group. In this section, we briefly review his construction and demonstrate, as Opfermann and Papadopoulos did [20], the existence of homogeneous HKT-connections.

Let G be a compact semi-simple Lie group. Let U be a maximal torus. Let $\mathfrak{g}$ and $\mathfrak{u}$ be their algebras. Choose a system of ordered roots with respect to $\mathfrak{u}_{\mathbf{C}}$. Let α_1 be a maximal positive root, and $\mathfrak{h}_1$ the dual space of α_1. Let ∂_1 be the $\mathfrak{sp}(1)$-subalgebra of $\mathfrak{g}$ such that its complexification is isomorphic to $\mathfrak{h}_1 \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1}$, where $\mathfrak{g}_{\alpha_1}$ and $\mathfrak{g}_{-\alpha_1}$ are the root spaces for α_1 and $-\alpha_1$ respectively. Let $\mathfrak{b}_1$ be the centralizer of ∂_1. Then there is a vector subspace $\mathfrak{f}_1$ composed of root spaces such that $\mathfrak{g} = \mathfrak{b}_1 \oplus \partial_1 \oplus \mathfrak{f}_1$. If $\mathfrak{b}_1$ is not Abelian, Joyce applies this decomposition to it. By inductively searching for $\mathfrak{sp}(1)$ subalgebras, he finds the following [19, Lemma 4.1].

Lemma 1. *The Lie algebra* $\mathfrak{g}$ *of a compact Lie group* G *decomposes as*

$$\mathfrak{g} = \mathfrak{b} \oplus_{j=1}^n \partial_j \oplus_{j=1}^n \mathfrak{f}_j, \tag{9}$$

with the following properties: (1) $\mathfrak{b}$ *is Abelian and* ∂_j *is isomorphic to* $\mathfrak{sp}(1)$. (2) $\mathfrak{b} \oplus_{j=1}^n \partial_j$ *contains* $\mathfrak{u}$. (3) *Set* $\mathfrak{b}_0 = \mathfrak{g}$, $\mathfrak{b}_n = \mathfrak{b}$ *and* $\mathfrak{b}_k = \mathfrak{b} \oplus_{j=k+1}^n \partial_j \oplus_{j=k+1}^n \mathfrak{f}_j$*. Then* $[\mathfrak{b}_k, \partial_j] = 0$ *for* $k \geq j$. (4) $[\partial_l, \mathfrak{f}_l] \subset \mathfrak{f}_l$. (5) *The adjoint representation of* ∂_l *on* $\mathfrak{f}_l$ *is reducible to a direct sum of the irreducible 2-dimensional representations of* $\mathfrak{sp}(1)$.

When the group G is semi-simple, the Killing-Cartan form is a negative definite inner product on the vector space $\mathfrak{g}$.

Lemma 2. *The Joyce Decomposition of a compact semi-simple Lie algebra is an orthogonal decomposition with respect to the Killing-Cartan form.*

Proof. Since Joyce Decomposition given as in (9) is inductively defined, it suffices to prove that the decomposition

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{b}_1 \oplus \partial_1 \oplus \mathfrak{f}_1 \tag{10}$$

is orthogonal. Recall that

$$\partial_1 = \langle \mathfrak{h}_1, X_{\alpha_1}, X_{-\alpha_1} \rangle, \ \mathfrak{f}_1 = \oplus_{\alpha_1 \neq \alpha > 0, \langle \alpha, \alpha_1 \rangle \neq 0} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha},$$
$$\mathfrak{b}_1 = \{h \in \mathfrak{u}_{\mathbf{C}} : \alpha_1(h) = 0\} \oplus_{\alpha_1 \neq \alpha > 0, \langle \alpha, \alpha_1 \rangle = 0} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}.$$

Since the Cartan subalgebra $\mathfrak{u}_{\mathbf{C}}$ is orthogonal to any root space, and it is an elementary fact that two root spaces $\mathfrak{g}_\alpha$, $\mathfrak{g}_\beta$ are orthogonal whenever $\alpha \neq \pm\beta$, $\mathfrak{f}_1$ is orthogonal to both $\mathfrak{b}_1$ and ∂_1. For the same reasons, ∂_1 is orthogonal to the summand $\oplus_{\alpha>0, \langle\alpha,\alpha_1\rangle=0} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ in $\mathfrak{b}_1$, and $\mathfrak{b}_1$ is orthogonal to the summand $\langle X_{\alpha_1}, X_{-\alpha_1} \rangle$ in ∂_1. Then ∂_1 is orthogonal to $\mathfrak{b}_1$ because for any element h in the Cartan subalgebra in $\mathfrak{b}_1$, $\langle h, h_1 \rangle = \alpha_1(h) = 0$. □

Let G be a compact semi-simple Lie group with rank r. Then

$$(2n - r)\mathfrak{u}(1) \oplus \mathfrak{g} \cong \mathbf{R}^n \oplus_{j=1}^n \partial_j \oplus_{j=1}^n \mathfrak{f}_j. \tag{11}$$

At the tangent space of the identity element of $T^{2n-r} \times G$, i.e. the Lie algebra $(2n - r)\mathfrak{u}(1) \oplus \mathfrak{g}$, a hypercomplex structure $\{I_1, I_2, I_3\}$ is defined as follows. Let $\{E_1, \ldots, E_n\}$ be a basis for $\mathbf{R}^n$. Choose isomorphisms ϕ_j from $\mathfrak{sp}(1)$, the real vector space of imaginary quaternions, to ∂_j. It gives a real linear identification from the quaternions $\mathbf{H}$ to $\langle E_j \rangle \oplus \partial_j$. If H_j, X_j and Y_j forms a basis for ∂_j such that $[H_j, X_j] = 2Y_j$ and $[H_j, Y_j] = -2X_j$, then

$$I_1 E_j = H_j, I_2 E_j = X_j, I_3 E_j = Y_j. \tag{12}$$

Define the action of I_a on $\mathfrak{f}_j$ by $I_a(v) = [v, \phi_j(\iota_a)]$, where $\iota_1 = i, \iota_2 = j, \iota_3 = k$. The complex structures $\{I_1, I_2, I_3\}$ at the other points of the group $T^{2n-r} \times G$ are obtained by left translations. These complex structures are integrable and form a hypercomplex structure [19].

Lemma 3. *When G is a compact semi-simple Lie group with rank r, there exists a negative definite bilinear form $\hat{B}$ on the decomposition $(2n - r)\mathfrak{u}(1) \oplus \mathfrak{g} \cong \mathbf{R}^n \oplus_{j=1}^n \partial_j \oplus_{j=1}^n \mathfrak{f}_j$ such that* (1) *its restriction to $\mathfrak{g}$ is the Killing-Cartan form,* (2) *it is hyper-Hermitian with respect to the hypercomplex structure, and* (3) *the above decomposition is orthogonal.*

Proof. In ∂_j, we choose an orthogonal basis $\{H_j, X_j, Y_j\}$ such that H_j is in the Cartan subalgebra and

$$B(H_j, H_j) = B(X_j, X_j) = B(Y_j, Y_j) = -\lambda_j^2. \tag{13}$$

On $\mathbf{R}^n = (2n - r)\mathfrak{u}(1) \oplus \mathfrak{b}$, choose $E_1, \ldots, E_n$ and extend the Killing-Cartan form so that

$$\hat{B}(E_i, E_j) = -\delta_{ij}\lambda_j^2. \tag{14}$$

It is now apparent that the extended Killing-Cartan form is hyper-Hermitian with respect to I_1, I_2 and I_3.

To show that the Killing-Cartan form is hyper-Hermitian on $\oplus_{j=1}^n \mathfrak{f}_j$, it suffices to verify that the Killing-Cartan form is hyper-Hermitian on $\mathfrak{f}_1$. It follows from the fact that $B(X, [Y, Z])$ is totally skew-symmetric with respect to X, Y, Z and the Jacobi identity. □

Let g be the left-translation of the extended Killing-Cartan form $-\hat{B}$. It is a bi-invariant metric on the manifold $T^{2n-r} \times G$. The Levi-Civita connection D is the bi-invariant connection. Let ∇ be the left-invariant connection defined by having all left-invariant vector fields being parallel. When X and Y are left-invariant vector fields

$$D_X Y = \frac{1}{2}[X, Y], \text{ and } \nabla_X Y = 0.$$

Since the hypercomplex structure and the hyper-Hermitian metric are left-invariant, the left-invariant connection is hyper-Hermitian. The torsion tensor for the left-invariant connection is $T(X, Y) = -[X, Y]$. The (3,0)-torsion tensor is

$$c(X, Y, Z) = -\hat{B}([X, Y], Z).$$

It is well known that c is a totally skew-symmetric 3-form. Therefore, the left-invariant connection is a HKT-structure on the group manifold $T^{2n-r} \times G$.

It is apparent that if one extends the Killing-Cartan form in an arbitrary way, then the resulting bi-invariant metric and left-invariant hypercomplex structure cannot make a hyper-Hermitian structure.

The above construction can be generalized to homogeneous spaces [20].

3. Characterization of HKT-Structures

In this section, we characterize HKT-structures in terms of the existence of a holomorphic object with respect to any complex structure in the hypercomplex structure. Through this characterization, we shall find other examples of HKT-manifolds. Toward the end of this section, we shall also reinterpret the twistor theory for HKT-geometry developed by Howe and Papadopoulos [17]. The results seem to indicate that the holomorphic characterization developed in the next paragraph will serve all the purposes that one wants the twistor theory of HKT-geometry to serve.

3.1. Holomorphic Characterization.

Proposition 2. *Let $(M, \mathcal{I}, g)$ be a hyper-Hermitian manifold and F_a be the Kähler form for (I_a, g). Then $(M, \mathcal{I}, g)$ is a HKT-structure if and only if $\partial_1(F_2 + iF_3) = 0$; or equivalently $\overline{\partial}_1(F_2 - iF_3) = 0$.*

Proof. Since $\partial_1(F_2 + iF_3) = \frac{1}{2}(dF_2 - d_1F_3) + \frac{i}{2}(d_1F_2 + dF_3)$, it is identically zero if and only if $d_1F_2 = -dF_3$, and $dF_2 = d_1F_3$.

Note that $F_2(I_1X, I_1Y) = g(I_2I_1X, I_1Y) = -g(I_2X, Y) = -F_2(X, Y)$. It follows that $d_1F_2 = (-1)^2 I_1 dI_1(F_2) = -I_1 dF_2$. As dF_2 is a 3-form, for any X, Y, Z tangent vectors,

$$\begin{aligned} -I_1dF_2(X, Y, Z) &= dF_2(I_1X, I_1Y, I_1Z) = dF_2(I_2I_3X, I_2I_3Y, I_2I_3Z) \\ &= -I_2dF_2(I_3X, I_3Y, I_3Z) = I_3I_2dF_2(X, Y, Z). \end{aligned}$$

Since F_2 is type (1,1) with respect to I_2, $I_2F_2 = F_2$. Then $d_1F_2 = -I_1dF_2 = I_3I_2dF_2 = I_3I_2dI_2F_2 = I_3d_2F_2$. On the other hand, $-dF_3 = I_3I_3dF_3 = I_3I_3dI_3F_3 = I_3d_3F_3$. Therefore, $d_2F_2 = d_3F_3$ if and only if $d_1F_2 = -dF_3$. Similarly, one can prove that $d_2F_2 = d_3F_3$ if and only if $d_1F_3 = dF_2$. It follows that $\partial_1(F_2 + iF_3) = 0$ if and only if $d_2F_2 = d_3F_3$. It is equivalent to $\nabla^2 = \nabla^3$, where ∇^a is the Bismut connection of the Hermitian structure (M, I_a, g). Since $I_1 = I_2I_3$, and $\nabla^2 = \nabla^3$, I_1 is parallel with respect to $\nabla^2 = \nabla^3$. By the uniqueness of the Bismut connection, $\nabla^1 = \nabla^2 = \nabla^3$. □

On any hypercomplex manifold $(M, \mathcal{I})$, if $F_2 - iF_3$ is a 2-form such that $-F_2(I_2X, Y) = g(X, Y)$ is positive definite and it is a non-holomorphic (0,2)-form with respect to I_1, then $(M, g, \mathcal{I})$ is a hyper-Hermitian manifold but it is not a HKT-structure. For example, a conformal change of a HKT-structure by a generic function gives a hyper-Hermitian structure which is not a HKT-structure so long as the dimension of the underlying manifold is at least eight. On the other hand Proposition 2 implies that every four-dimensional hyper-Hermitian manifold is a HKT-structure, a fact also proven in [13, Sect. 2.2].

In the proof of Proposition 2, we also derive the following [17].

Corollary 1. *Suppose F_1, F_2 and F_3 are the Kähler forms of a hyper-Hermitian structure. Then the hyper-Hermitian structure is a HKT-structure if and only if*

$$d_i F_j = -2\delta_{ij} c - \epsilon_{ijk} dF_k. \tag{15}$$

Theorem 1. *Let $(M, \mathcal{I})$ be a hypercomplex manifold and $F_2 - iF_3$ be a* (0,2)*-form with respect to I_1 such that $\overline{\partial}_1(F_2 - iF_3) = 0$ or equivalently $\partial_1(F_2 + iF_3) = 0$ and $-F_2(I_2X, Y) = g(X, Y)$ is a positive definite symmetric bilinear form. Then $(M, \mathcal{I}, g)$ is a HKT-structure.*

Proof. In view of the last proposition, it suffices to prove that the metric g along with the given hypercomplex structure $\mathcal{I}$ is hyper-Hermitian.

Note that $F_2 - iF_3$ is type (0,2) with respect to I_1. Since $X - iI_1X$ is a type (1,0)-vector with respect to I_1, $(F_2 - iF_3)(X - iI_1X, Y) = 0$ for any vectors X and Y. It is equivalent to the identity $F_2(I_1X, Y) = -F_3(X, Y)$. Then

$$F_3(I_3X, Y) = -F_2(I_1I_3X, Y) = F_2(I_2X, Y) = -g(X, Y).$$

So $F_3(I_3X, I_3Y) = F_3(X, Y)$, and g is Hermitian with respect to I_3. Since the metric g is Hermitian with respect to I_2 and $I_1 = I_2I_3$, g is also Hermitian with respect to I_1. □

3.2. HKT-Structures on Compact Nilmanifolds. In this section, we apply the last theorem to construct a homogeneous HKT-structure on some compact nilmanifolds.

Let $\{X_1, ..., X_{2n}, Y_1, ..., Y_{2n}, Z\}$ be a basis for $\mathbf{R}^{4n+1}$. Define commutators by: $[X_i, Y_i] = Z$, and all others are zero. These commutators define on $\mathbf{R}^{4n+1}$ the structure of the *Heisenberg Lie algebra* h_{2n}. Let $\mathbf{R}^3$ be the 3-dimensional Abelian algebra. The direct sum $\mathbf{n} = h_{2n} \oplus \mathbf{R}^3$ is a 2-step nilpotent algebra whose center is four-dimensional. Fix a basis $\{E_1, E_2, E_3\}$ for $\mathbf{R}^3$ and consider the following endomorphisms of $\mathbf{n}$ [8]:

$$\begin{aligned} &I_1 \;:\; X_i \to Y_i, Z \to E_1, E_2 \to E_3; \\ &I_2 \;:\; X_{2i+1} \to X_{2i}, Y_{2i-1} \to Y_{2i}, Z \to E_2, E_1 \to E_3; \\ &I_1^2 = I_2^2 = -\text{identity}, \qquad\qquad I_3 = I_1 I_2. \end{aligned}$$

Clearly $I_1I_2 = -I_2I_1$. Moreover, for $a = 1, 2, 3$ and $X, Y \in \mathbf{n}$, $[I_aX, I_aY] = [X, Y]$ so I_a are Abelian complex structures on $\mathbf{n}$ in the sense of [1] and in particular are integrable. It implies that $\{I_a : a = 1, 2, 3\}$ is a left invariant hypercomplex structure on the simply connected Lie group N whose algebra is $\mathbf{n}$. It is known that the complex structures I_a on $\mathbf{n}$ satisfy:

$$d(\Lambda^{1,0}_{I_a}\mathbf{n}^*) \in \Lambda^{1,1}_{I_a}\mathbf{n}^*,$$

where $\mathbf{n}^*$ is the space of left invariant 1-forms on N and $\Lambda^{i,j}_{I_a}\mathbf{n}^*$ is the (i,j)-component of $\mathbf{n}^* \otimes \mathbf{C}$ with respect to I_a [25]. But then we have $d(\Lambda^{2,0}_{I_a}\mathbf{n}^*) \in \Lambda^{2,1}_{I_a}\mathbf{n}^*$ and any left invariant (2,0)-form is ∂_1-closed. Now consider the invariant metric on N for which the basis $\{X_i, Y_i, Z, E_a\}$ is orthonormal. Since it is compatible with the structures I_a in view of Theorem 1 we obtain a left-invariant HKT-structure on N. Noting that N is isomorphic to the product $H_{2n} \times \mathbf{R}^3$ of the Heisenberg Lie group H_{2n} and the Abelian group $\mathbf{R}^3$ we have:

Corollary 2. *Let Γ be a cocompact lattice in the Heisenberg group H_{2n} and $\mathbf{Z}^3$ a lattice in $\mathbf{R}^3$. The compact nilmanifold $(\Gamma \times \mathbf{Z}^3)\backslash N$ admits a HKT-structure.*

3.3. Twist of Hyper-Kähler Manifolds with Torsions. Suppose that $(M, \mathcal{I})$ is a hypercomplex manifold, a $\mathrm{U}(1)$-instanton P is a principal $\mathrm{U}(1)$-bundle with a $\mathrm{U}(1)$-connection 1-form θ such that its curvature 2-form is type-(1,1) with respect to every complex structure in $\mathcal{I}$ [7,10]. Let $\Psi_M : \mathrm{U}(1) \to \mathrm{Aut}(M)$ be a group of hypercomplex automorphism, and let $\Psi_P : \mathrm{U}(1) \to \mathrm{Aut}(P)$ be a lifting of Ψ_M. Let $\Phi : \mathrm{U}(1) \to \mathrm{Aut}\, P$ be the principal $U(1)$-action on the bundle P, and $\triangle(g)$ be the diagonal product $\Phi(g)\Psi_P(g)$ action on P. A theorem of Joyce [19, Theorem 2.2] states that the quotient space $W = P/\triangle(\mathrm{U}(1))$ of the total space of P with respect to the diagonal action $\triangle$ is a hypercomplex manifold whenever the vector field generated by $\triangle(\mathrm{U}(1))$ transversal to the horizontal distribution of the connection θ. The quotient space W is called a twist of the hypercomplex manifold M.

Now suppose that $(M, \mathcal{I}, g)$ is a HKT-structure and P is a $\mathrm{U}(1)$-instanton with connection form θ. Suppose that $\Psi_M : \mathrm{U}(1) \to \mathrm{Aut}(M)$ is a group of hypercomplex isometry. Due to the uniqueness of HKT-structure, Ψ_M is a group of automorphism of the HKT-structure.

Corollary 3. *The twist manifold W admits a HKT-structure.*

Proof. Let $\phi : P \to M$ and $\Delta : P \to W$ be the projections from the instanton bundle P to M and the twist W respectively. The connection θ defines a splitting of the tangent bundle of P into horizontal and vertical components: $TP = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{H} = Ker\theta$. We define endomorphisms $\tilde{I}_a$ on TP as follows: $\tilde{I}_a = 0$ on vertical directions, and when $\tilde{v}$ is a horizontal lift of a tangent vector v to M, define $\tilde{I}_a\tilde{v} = \widetilde{I_a v}$.

Since the fibers of the projection Δ are transversal to the horizontal distribution, for any tangent vector $\hat{v}$ to W, there exists a horizontal vector $\tilde{v}$ such that $d\Delta\tilde{v} = \hat{v}$. Define $\hat{I}_a$ and $\hat{g}$ on W by $\hat{I}_a\hat{v} = d\Delta(\tilde{I}_a\tilde{v})$ and $\hat{g}(\hat{v}, \hat{w}) = \tilde{g}(\tilde{v}, \tilde{w})$. As the diagonal action is a group of hyper-holomorphic isometries, the almost complex structures $\hat{I}_a$ and metric $\hat{g}$ are well defined.

To verify that $\hat{I}_a$ are integrable complex structures on W, we first observe that: for horizontal vector fields X and Y, $d\Delta[X, Y] = [d\Delta X, d\Delta Y]$, $d\phi[X, Y] = [d\phi X, d\phi Y]$ and $d\Delta\tilde{I}_a = \hat{I}_a d\Delta$, $d\phi\tilde{I}_a = I_a d\phi$. Through these relations, we establish the following relations between Nijenhius tensors of I_a, $\hat{I}_a$ and $\tilde{I}_a$:

$$d\Delta\tilde{N}_a(X, Y) = \hat{N}_a(d\Delta X, d\Delta Y) \qquad \text{and} \qquad d\phi\tilde{N}_a(X, Y) = N_a(d\phi X, d\phi Y).$$

The second identity implies that the horizontal part of $\tilde{N}_a(X, Y)$ vanishes because the complex structures I_a are integrable. With the first identity, it follows that the Nijenhius tensor for $\hat{I}_a$ vanishes if the vertical part of $\tilde{N}_a(X,Y)$ also vanishes. To calculate the vertical part, we have

$$\begin{aligned}\theta(\tilde{N}_a(X,Y)) &= \frac{1}{4}\theta([X,Y]+\tilde{I}_a[\tilde{I}_aX,Y]+\tilde{I}_a[X,\tilde{I}_aY]-[\tilde{I}_aX,\tilde{I}_aY]) \\ &= \frac{1}{4}\theta([X,Y]-[\tilde{I}_aX,\tilde{I}_aY]) = \frac{1}{4}(d\theta(X,Y)-d\theta(I_aX,I_aY)).\end{aligned}$$

Since θ is an instanton, $d\theta(X,Y)-d\theta(I_aX,I_aY)=0$. It follows that $\hat{I}_a$ are integrable.

To check that $\hat{g}$ is a HKT-metric, we first observe that $d\Delta$ and $d\phi$ give rise to isomorphisms of $\Lambda^{(p,q)}M$, $\Lambda^{(p,q)}\mathcal{H}$ and $\Lambda^{(p,q)}W$ when we fix the structures I_1, $\hat{I}_1$ and $\tilde{I}_1$. Let the Kähler forms of the structures I_a and $\hat{I}_a$ be denoted by F_a and $\hat{F}_a$ respectively. Now if X,Y and Z are sections of $\mathcal{H}^{(1,0)}$ then

$$X(\Delta^*(\hat{F}_2+i\hat{F}_3))(Y,Z) = X(\phi^*(F_2+iF_3))(Y,Z).$$

Since $d\theta$ is type (1,1), $\theta([X,Y]) = d\theta(X,Y) = 0$. It means that $[X,Y]$ is a section of $\mathcal{H}^{(1,0)}$. Therefore, $\Delta^*(\hat{F}_2+i\hat{F}_3)([X,Y],Z) = \phi^*(F_2+iF_3)([X,Y],Z)$. It follows that

$$(\Delta^* d(\hat{F}_2+i\hat{F}_3))|_{\Lambda^{(3,0)}\mathcal{H}} = (d\Delta^*(\hat{F}_2+i\hat{F}_3))|_{\Lambda^{(3,0)}\mathcal{H}} = d\phi^*(F_2+iF_3))|_{\Lambda^{(3,0)}\mathcal{H}} = 0.$$

Hence $d(\hat{F}_2+i\hat{F}_3)|_{\Lambda^{(3,0)}W}=0$ and the corollary follows from Proposition 2. □

3.4. Twistor Theory of HKT-Geometry. When $(M,\mathcal{I})$ is a 4n-dimensional hypercomplex manifold, the smooth manifold $Z=M\times S^2$ admits an integrable complex structure. It is defined as follows. For a unit vector $\boldsymbol{a}=(a_1,a_2,a_3)\in\mathbf{R}^3$, let $I_{\boldsymbol{a}}$ be the complex structure $a_1I_1+a_2I_2+a_3I_3$ in the hypercomplex structure $\mathcal{I}$. Let $J_{\boldsymbol{a}}$ be the complex structure on S^2 defined by cross product in $\mathbf{R}^3$: $J_{\boldsymbol{a}}\boldsymbol{w}=\boldsymbol{a}\times\boldsymbol{w}$. Then the complex structure on $Z=M\times S^2$ at the point $(x,\boldsymbol{a})$ is $\mathcal{J}_{(x,\boldsymbol{a})}=I_{\boldsymbol{a}}\oplus J_{\boldsymbol{a}}$. It is well known from twistor theory that this complex structure is integrable [24]. We shall have to consider a non-integrable almost complex structure $\mathcal{J}_2=I\oplus(-J)$. Unless specified otherwise, we discuss holomorphicity on Z in terms of the integrable complex structure $\mathcal{J}$.

With respect to $\mathcal{J}$, the fibers of the projection π from $Z=M\times S^2$ onto its first factor are holomorphic curves with genus zero. It can be proved that the holomorphic normal bundles are $\oplus^{2n}\mathcal{O}(1)$. The antipodal map τ on the second factor is an anti-holomorphic map on the twistor space Z leaving the fibers of the projection π invariant.

The projection p onto the second smooth factor of $Z=M\times S^2$ is a holomorphic map such that the inverse image of a point (a_1,a_2,a_3) is the manifold M equipped with the complex structure $a_1I_1+a_2I_2+a_3I_3$. If $\mathcal{D}$ is the sheaf of kernel of the differential dp, then we have the exact sequence

$$0\to\mathcal{D}\to\Theta_Z\xrightarrow{dp}p^*\Theta_{\mathbf{CP}^1}\to 0. \tag{16}$$

Real sections, i.e. τ-invariant sections, of the holomorphic projection p are fibers of the projection from Z onto M.

Twistor theory shows that there is a one-to-one correspondence between the hypercomplex manifold $(M,\mathcal{I})$ and its twistor space Z with the complex structure $\mathcal{J}$, the anti-holomorphic map τ, the holomorphic projection p and the sections of the projection p with prescribed normal bundle [21].

It is not surprising that when a hypercomplex manifold has a HKT-structure, there is an additional geometric structure on the twistor space. The following theorem is essentially developed in [17].

Theorem 2. *Let $(M, \mathcal{I}, g)$ be a 4n-dimensional HKT-structure. Then the twistor space Z is a complex manifold such that*

1. *the fibers of the projection $\pi : Z \to M$ are rational curves with holomorphic normal bundle $\oplus^{2n}\mathcal{O}(1)$,*
2. *there is a holomorphic projection $p : Z \to \mathbf{CP}^1$ such that the fibers are the manifold M equipped with complex structures of the hypercomplex structure $\mathcal{I}$,*
3. *there is a $\mathcal{J}_2$-holomorphic section of $\wedge^{(0,2)}\mathcal{D} \otimes p^*\overline{\Theta}_{\mathbf{CP}^1}$ defining a positive definite (0,2)-form on each fiber,*
4. *there is an anti-holomorphic map τ compatible with 1, 2 and 3 and inducing the antipodal map on $\mathbf{CP}^1$.*

Conversely, if Z is a complex manifold with a non-integrable almost complex structure J_2 with the above four properties, then the parameter space of real sections of the projection p is a 4n-dimensional manifold M with a natural HKT-structure for which Z is the twistor space.

Proof. Given a HKT-structure, then only Part 3 in the first half of this theorem is a new observation. It is a generalization of Theorem 1. Through the stereographic projection,

$$\zeta \mapsto \boldsymbol{a} = \frac{1}{1+|\zeta|^2}\left(1-|\zeta|^2, -i(\zeta-\overline{\zeta}), -(\zeta+\overline{\zeta})\right), \tag{17}$$

ζ is a complex coordinate of the Riemann sphere. Note that

$$\frac{1}{1+|\zeta|^2}\begin{pmatrix} 1-|\zeta|^2 & i(\zeta-\overline{\zeta}) & \zeta+\overline{\zeta} \\ -i(\zeta-\overline{\zeta}) & 1+\frac{1}{2}(\zeta^2+\overline{\zeta}^2) & -\frac{i}{2}(\zeta^2-\overline{\zeta}^2) \\ -(\zeta+\overline{\zeta}) & -\frac{i}{2}(\zeta^2-\overline{\zeta}^2) & 1-\frac{1}{2}(\zeta^2+\overline{\zeta}^2) \end{pmatrix}$$

is a special orthogonal matrix. Let $\boldsymbol{b}$ and $\boldsymbol{c}$ be the second and third column vectors respectively. Consider the complex structure

$$I_{\boldsymbol{a}} = \frac{1}{1+|\zeta|^2}\left((1-|\zeta|^2)I_1 - i(\zeta-\overline{\zeta})I_2 - (\zeta+\overline{\zeta})I_3\right).$$

According to Theorem 1, the 2-form

$$F_{\boldsymbol{b}} - iF_{\boldsymbol{c}} = \frac{1}{1+|\zeta|^2}\Big((F_2 - iF_3) - 2i\overline{\zeta}F_1 + \overline{\zeta}^2(F_2+iF_3)\Big) \tag{18}$$

is holomorphic with respect to $I_{\boldsymbol{a}}$.

Due to the integrability of the complex structure $I_{\boldsymbol{a}}$, $d_{\boldsymbol{a}}$ is linear in $\boldsymbol{a}$. Therefore,

$$d_{\boldsymbol{a}} = \frac{1}{1+|\zeta|^2}\left((1-|\zeta|^2)d_1 - i(\zeta-\overline{\zeta})d_2 - (\zeta+\overline{\zeta})d_3\right). \tag{19}$$

Note that $\overline{\zeta}$ is holomorphic with respect to the almost complex structure $\mathcal{J}_2$. More precisely, consider the $\overline{\partial}$-operator with respect to the almost complex structure $\mathcal{J}_2$: on n-forms, it is

$$\overline{\delta} = \frac{1}{2}(d - i(-1)^n \mathcal{J}_2 d \mathcal{J}_2), \tag{20}$$

then $\mathcal{J}_2 d\overline{\zeta} = i\overline{\zeta}$, and $\overline{\delta\zeta} = 0$. It follows that at $(x, \boldsymbol{a})$ on $Z = M \times S^2$,

$$\begin{aligned}
&\overline{\delta}\left(-2i\overline{\zeta}F_1 + (1+\overline{\zeta}^2)F_2 - i(1-\overline{\zeta}^2)F_3\right) \\
&= -2i\overline{\zeta\delta}F_1 + (1+\overline{\zeta}^2)\overline{\delta}F_2 - i(1-\overline{\zeta}^2)\overline{\delta}F_3 \\
&= -2i\overline{\zeta\partial}_{\boldsymbol{a}}F_1 + (1+\overline{\zeta}^2)\overline{\partial}_{\boldsymbol{a}}F_2 - i(1-\overline{\zeta}^2)\overline{\partial}_{\boldsymbol{a}}F_3 \\
&= \frac{1}{2}\left(-2i\overline{\zeta}dF_1 + (1+\overline{\zeta}^2)dF_2 - i(1-\overline{\zeta}^2)dF_3\right) \\
&\quad - \frac{i}{2}\left(-2i\overline{\zeta}d_{\boldsymbol{a}}F_1 + (1+\overline{\zeta}^2)d_{\boldsymbol{a}}F_2 - i(1-\overline{\zeta}^2)d_{\boldsymbol{a}}F_3\right).
\end{aligned}$$

Now (19) and (15) together imply that the twisted 2-form $(F_2 - iF_3) - 2i\overline{\zeta}F_1 + \overline{\zeta}^2(F_2 + iF_3)$ is closed with respect to $\overline{\delta}$. Therefore, it is a $\mathcal{J}_2$-holomorphic section.

Since ζ is a holomorphic coordinate on S^2, the homogeneity shows that this section is twisted by $\overline{\mathcal{O}(2)}$.

The inverse construction is a consequence of the inverse construction of hypercomplex manifold [21] and Theorem 1. □

As the almost complex structure $\mathcal{J}_2$ is never integrable [9], twistor theory loses substantial power of holomorphic geometry when we study HKT-structure. Therefore, we focus on the application of Theorem 1.

4. Potential Theory

Theorem 1 shows that the form $F_2 + iF_3$ is a ∂_1-closed (2,0)-form on a HKT-manifold. It is natural to consider a differential form β_1 as potential 1-form for $F_2 + iF_3$ if $\partial_1\beta_1 = F_2 + iF_3$. A priori, the 1-form β_1 depends on the choice of the complex structure I_1. The potential 1-form for $F_3 + iF_1$, if it exists, depends on I_2, and so on. In this section, we seek a function that generates all Kähler forms.

4.1. Potential Functions. A function μ is a potential function for a hyper-Kähler manifold $(M, \mathcal{I}, g)$ if the Kähler forms F_a are equal to $dd_a\mu$. Since $d_a = (-1)^n I_a d I_a$ on n-forms, $d_a\mu = I_a d\mu$. Therefore,

$$\begin{aligned}
d_1 d_2\mu &= d_1 I_2 d\mu = -I_1 d I_1 I_2 d\mu = -I_1 d I_3 d\mu = -I_1 dd_3\mu \\
&= -I_1 \Omega_3 = \Omega_3 = dd_3\mu.
\end{aligned}$$

Now we generalize this concept to HKT-manifolds.

Definition 3. *Let $(M, \mathcal{I}, g)$ be a HKT-structure with Kähler forms F_1, F_2 and F_3. A possibly locally defined function μ is a potential function for the HKT-structure if*

$$F_1 = \frac{1}{2}(dd_1 + d_2d_3)\mu, \quad F_2 = \frac{1}{2}(dd_2 + d_3d_1)\mu, \quad F_3 = \frac{1}{2}(dd_3 + d_1d_2)\mu. \tag{21}$$

Due to the identities $dd_a + d_a d = 0$ and $d_a d_b + d_b d_a = 0$, μ is a potential function if and only if

$$F_{\boldsymbol{a}} = \frac{1}{2}(dd_{\boldsymbol{a}} + d_{\boldsymbol{b}} d_{\boldsymbol{c}})\mu,$$

when $\boldsymbol{a} = \boldsymbol{b} \times \boldsymbol{c}$ and $F_{\boldsymbol{a}}$ is the Kähler form for the complex structure $I_{\boldsymbol{a}} = a_1 I_1 + a_2 I_2 + a_3 I_3$. Moreover, the torsion 3-form is equal to $-\frac{1}{4} d_1 d_2 d_3 \mu$. Furthermore, since $\partial_a = \frac{1}{2}(d + i d_a)$ and $\overline{\partial}_a = \frac{1}{2}(d - i d_a)$,

$$F_2 + iF_3 = \frac{1}{2}(dd_2 + idd_3 + id_1 d_2 - d_1 d_3)\mu = 2\partial_1 I_2 \overline{\partial}_1 \mu. \tag{22}$$

Conversely, if a function μ satisfies the above identity, it satisfies the last two identities in (21). Since the metric is hyper-Hermitian, for any vectors X and Y, $F_1(X,Y) = F_2(I_3 X, Y)$. Through the integrability of the complex structures I_1, I_2, I_3, the quaternion identities (5) and the last two identities in (21), one derives the first identity in (21). Therefore, we have the following theorem which justifies our definition for potential functions.

Theorem 3. *Let $(M, \mathcal{I}, g)$ be a HKT-structure with Kähler form F_1, F_2 and F_3. A possibly locally defined function μ is a potential function for the HKT-structure if*

$$F_2 + iF_3 = 2\partial_1 I_2 \overline{\partial}_1 \mu. \tag{23}$$

In this context, a HKT-structure is hyper-Kähler if and only if the potential function satisfies the following identities:

$$dd_1\mu = d_2 d_3 \mu, \quad dd_2 \mu = d_3 d_1 \mu, \quad dd_3 \mu = d_1 d_2 \mu. \tag{24}$$

Corollary 4. *Every hypercomplex manifold locally admits a HKT-metric.*

Proof. We fix the complex structure I_2 and consider a locally defined Kähler potential function μ with respect to I_2. Then $dd_2\mu(X, I_2 X) > 0$ for every nonzero X. Simple calculation shows that if $Y = I_3 X$ then

$$(dd_2 + d_3 d_1)\mu(X, I_2 X) = dd_2\mu(X, I_2 X) + dd_2 \mu(Y, I_2 Y) > 0. \tag{25}$$

Then we see that the form $F_2 + iF_3 = 2\partial_1 I_2 \overline{\partial}_1 \mu$ satisfies the conditions of Theorem 1 and hence is a local HKT-potential function, thus defining a HKT-metric. □

Remark. As in the Kähler case, compact manifolds do not admit globally defined HKT potential. To verify, let f be a potential function and g be the corresponding induced metric. Define the complex Laplacian of f with respect to g:

$$\overline{\partial}^* \overline{\partial} f = \triangle^c f = g(dd_1 f, F_1).$$

Then $0 \leq 2g(F_1, F_1) = g(dd_1 f + d_2 d_3 f, F_1) = 2\triangle^c f$, because

$$g(d_2 d_3 f, F_1) = g(-I_2 dd_1 f, F_1) = -g(dd_1 f, I_2 F_1) = g(dd_1 f, F_1) = \triangle^c f.$$

Now the remark follows from the standard arguments involving the maximum principle for second order elliptic differential equation just like in the Kähler case since $\triangle^c f$ does not have zero-order terms.

Remark. If we introduce the following quaternionic operators acting on quaternionic valued forms on the left: $\partial^H = d + id_1 + jd_2 + kd_3$, and $\overline{\partial}^H = d - id_1 - jd_2 - kd_3$, then a real-valued function μ is a HKT-potential if $\partial^H \overline{\partial}^H \mu = -2iF_1 - 2jF_2 - 2kF_3$.

If we identify $\mathbf{H}^n$ with $\mathbf{C}^{2n}$, we deduce like in Corollary 4 that any pluri-subharmonic function in the domain of $\mathbf{C}^{2n}$ is a HKT-potential. The converse however is wrong. As we shall see in 4.3 the function $\log(|z|^2 + |w|^2)$ is a HKT potential in $\mathbf{C}^{2n}\backslash\{0\}$ but is not pluri-subharmonic.

Remark. Given a HKT-metric g with Kähler forms F_1, F_2 and F_3, for any real-valued function μ we consider

$$\hat{F}_2 + i\hat{F}_3 = F_2 + iF_3 + \partial_1 I_2 \overline{\partial}_1 \mu.$$

According to Theorem 1 and other results in this section, whenever the form $\hat{g}(X, Y) := -\hat{F}_2(I_2 X, Y)$ is positive definite, we obtain a new HKT-metric with respect to the old hypercomplex structure.

4.2. Transformations of HKT-Potentials. Let $(M, \mathcal{I}, g)$ be a HKT-manifold with potential function μ. The Kähler forms are given by $\Omega_a = \frac{1}{2}(dd_a + d_b d_c)\mu$. We consider HKT-structures generated by potential functions through μ.

Theorem 4. *Suppose $(M, \mathcal{I}, g)$ is a HKT-manifold with a potential function μ. For any smooth function f of one variable, let U be the open subset of M on which μ is defined and*

$$f'(\mu) + \frac{1}{4} f''(\mu) |\nabla \mu|^2 > 0. \tag{26}$$

Define a symmetric bilinear form $\hat{g}$ by

$$\hat{g} = f'(\mu) g + \frac{1}{4} f''(\mu)(d\mu \otimes d\mu + I_1 d\mu \otimes I_1 d\mu + I_2 d\mu \otimes I_2 d\mu + I_3 d\mu \otimes I_3 d\mu). \tag{27}$$

Then $(U, \mathcal{I}, \hat{g})$ is a HKT-structure with $f(\mu)$ as its potential.

Proof. Since μ is a HKT-potential for the HKT-structure $(\mathcal{I}, g)$, $\Omega_2 + i\Omega_3 = 2\partial_1 I_2 \overline{\partial}_1 \mu$. It follows that

$$\begin{aligned} 2\partial_1 I_2 \overline{\partial}_1 f &= 2\partial_1 f'(\mu) I_2 \overline{\partial}_1 \mu = 2f'(\mu)\partial_1 I_2 \overline{\partial}_1 \mu + 2f''(\mu)\partial_1 \mu \wedge I_2 \overline{\partial}_1 \mu \\ &= f'(\mu)(\Omega_2 + i\Omega_3) + \frac{1}{2} f''(\mu)(d\mu + id_1\mu) \wedge (I_2 d\mu - iI_2 d_1 \mu). \end{aligned}$$

When F_2 and F_3 are the real and imaginary parts of $2\partial_1 I_2 \overline{\partial}_1 f$ respectively, then

$$F_2 = f'(\mu)\Omega_2 + \frac{1}{2} f''(\mu)(d\mu \wedge I_2 d\mu + d_1 \mu \wedge I_2 d_1 \mu). \tag{28}$$

It is now straightforward to verify that $-\hat{F}_2(I_2 X, Y) = \hat{g}(X, Y)$. Therefore, $\hat{g}$ together with given hypercomplex structure defines a HKT-structure with the function f as its potential so long as $\hat{g}$ is positive definite.

Since g is hyper-Hermitian, the vector fields $Y_0 = \nabla\mu$ and $Y_a = I_a\nabla\mu$ are mutually orthogonal with equal length. At any point where Y_0 is not the zero vector, we extend $\{Y_0, Y_1, Y_2, Y_3\}$ to an orthonormal frame with respect to the hyper-Kähler metric g. Any vector X can be written as $X = a_0Y_0 + a_1Y_1 + a_2Y_2 + a_3Y_3 + X^\perp$, where $X^\perp$ is in the orthogonal complement of $\{Y_0, Y_1, Y_2, Y_3\}$. Note that

$$d\mu(X^\perp) = g(\nabla\mu, X^\perp) = 0, \text{ and}$$
$$I_a d\mu(X^\perp) = -g(\nabla\mu, I_aX^\perp) = g(I_a\nabla\mu, X^\perp) = 0.$$

Also, for $1 \le a \ne b \le 3$,

$$d\mu(Y_a) = g(\nabla\mu, I_a\nabla\mu) = 0, \quad d\mu(Y_0) = |\nabla\mu|^2,$$
$$I_b d\mu(Y_a) = -g(\nabla\mu, I_bI_a\nabla\mu) = 0, \quad I_a d\mu(Y_a) = -g(\nabla\mu, I_a^2\nabla\mu) = |\nabla\mu|^2.$$

Then

$$\hat g(X,X) = f'(\mu)(\sum_{\ell=0}^{3} a_\ell^2)|\nabla\mu|^2 + \frac{f''(\mu)}{4}(\sum_{\ell=0}^{3} a_\ell^2)|\nabla\mu|^4$$
$$= (f'(\mu) + \frac{f''(\mu)}{4}|\nabla\mu|^2)(\sum_{\ell=0}^{3} a_\ell^2)|\nabla\mu|^2.$$

Therefore, $\hat g$ is positive definite on the open set defined by the inequality (26). □

Note that for any positive integer m, $f(\mu) = \mu^m$ satisfies (26) whenever μ is positive. So does $f(\mu) = e^\mu$. Therefore, if g is a HKT-metric with a positive potential function μ, the following metrics are HKT-metrics:

$$g_m = m\mu^{m-2}(\mu g + \frac{m-1}{4}\Big(d\mu\otimes d\mu + I_1d\mu\otimes I_1d\mu$$
$$+I_2d\mu\otimes I_2d\mu + I_3d\mu\otimes I_3d\mu)\Big),$$
$$g_\infty = e^\mu(g + \frac{1}{4}(d\mu\otimes d\mu + I_1d\mu\otimes I_1d\mu + I_2d\mu\otimes I_2d\mu + I_3d\mu\otimes I_3d\mu)).$$

4.3. Inhomogeneous HKT-Structures on $S^1\times S^{4n-3}$. On the complex vector space $(\mathbf{C}^n\oplus\mathbf{C}^n)\backslash\{0\}$, let (z_α, w_α), $1\le\alpha\le n$, be its coordinates. We define a hypercomplex structure to contain this complex structure as follows:

$$\begin{array}{llll} I_1dz_\alpha = -idz_\alpha, & I_1dw_\alpha = -idw_\alpha, & I_1d\overline{z}_\alpha = id\overline{z}_\alpha, & I_1d\overline{w}_\alpha = id\overline{w}_\alpha,\\ I_2dz_\alpha = d\overline{w}_\alpha, & I_2dw_\alpha = -d\overline{z}_\alpha, & I_2d\overline{z}_\alpha = dw_\alpha, & I_2d\overline{w}_\alpha = -dz_\alpha,\\ I_3dz_\alpha = id\overline{w}_\alpha, & I_3dw_\alpha = -id\overline{z}_\alpha, & I_3d\overline{z}_\alpha = -idw_\alpha, & I_3d\overline{w}_\alpha = idz_\alpha. \end{array}$$

The function $\mu = \frac{1}{2}(|z|^2+|w|^2)$ is the hyper-Kähler potential for the standard Euclidean metric:

$$g = \frac{1}{2}(dz_\alpha\otimes d\overline{z}_\alpha + d\overline{z}_\alpha\otimes dz_\alpha + dw_\alpha\otimes d\overline{w}_\alpha + d\overline{w}_\alpha\otimes dw_\alpha). \tag{29}$$

Since $|\nabla\mu|^2 = 2\mu$, the function $f(\mu) = \ln\mu$ satisfies the inequality (26) on $\mathbf{C}^{2n}\backslash\{0\}$. By Theorem 4, $\ln\mu$ is the HKT-potential for a HKT-metric $\hat g$ on $\mathbf{C}^{2n}\backslash\{0\}$.

Next for any real number r, with $0 < r < 1$, and $\theta_1, \ldots, \theta_n$ modulo 2π, we consider the integer group $\langle r \rangle$ generated by the following action on $(\mathbf{C}^n \oplus \mathbf{C}^n)\backslash\{0\}$:

$$(z_\alpha, w_\alpha) \mapsto (re^{i\theta_\alpha} z_\alpha, re^{-i\theta_\alpha} w_\alpha). \tag{30}$$

One can check that the group $\langle r \rangle$ is a group of hypercomplex transformations. As observed in [22], the quotient space of $(\mathbf{C}^n \oplus \mathbf{C}^n)\backslash\{0\}$ with respect to $\langle r \rangle$ is the manifold $S^1 \times S^{4n-1} = S^1 \times \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$. Since the group $\langle r \rangle$ is also a group of isometries with respect to the HKT-metric $\hat{g}$ determined by $f(\mu) = \ln \mu$, the HKT-structure descends from $(\mathbf{C}^n \oplus \mathbf{C}^n)\backslash\{0\}$ to a HKT-structure on $S^1 \times S^{4n-1}$. Since the hypercomplex structures on $S^1 \times S^{4n-1}$ are parametrized by $(r, \theta_1, \ldots, \theta_n)$ and a generic hypercomplex structure in this family is inhomogeneous [22], we obtain a family of inhomogeneous HKT-structures on the manifold $S^1 \times S^{4n-1}$.

Theorem 5. *Every hypercomplex deformation of the homogeneous hypercomplex structure on $S^1 \times S^{4n-1}$ admits a HKT-metric.*

Furthermore, $\hat{F}_2 + i\hat{F}_3 = 2\partial_1 I_2 \overline{\partial}_1 \mu$ descends to $S^1 \times S^{4n-1}$. However, the function μ does not descend to $S^1 \times S^{4n-1}$. Therefore, this (2,0)-form has a potential form $I_2\overline{\partial}_1\mu$ but not a globally defined potential function.

4.4. Associated Bundles of Quaternionic Kähler Manifolds. When M is a quaternionic Kähler manifold, i.e. the holonomy of the Riemannian metric is contained in the group $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, the representation of $\mathrm{Sp}(1)$ on quaternions $\mathbf{H}$ defines an associated fiber bundle $\mathcal{U}(M)$ over the smooth manifold M with $\mathbf{H}\backslash\{0\}/\mathbf{Z}_2$ as fiber. Swann finds that there is a hyper-Kähler metric g on $\mathcal{U}(M)$ whose potential function μ is the length of the radius coordinate vector field along each fiber [27]. As in the last example, $\ln \mu$ is the potential function of a HKT-structure with metric $\hat{g}$.

Again, metric $\hat{g}$ and the hypercomplex structure are both invariant of fiberwise real scalar multiplication. Therefore, the HKT-structure with metric $\hat{g}$ descends to the compact quotients defined by integer groups generated by fiberwise real scalar multiplications.

5. Reduction

First of all, we recall the construction of hypercomplex reduction developed by Joyce [18]. Let G be a compact group of hypercomplex automorphisms on M. Denote the algebra of hyper-holomorphic vector fields by $\mathfrak{g}$. Suppose that $\nu = (\nu_1, \nu_2, \nu_3) : M \longrightarrow \mathbf{R}^3 \otimes \mathfrak{g}$ is a G-equivariant map satisfying the following two conditions. The Cauchy-Riemann condition: $I_1 d\nu_1 = I_2 d\nu_2 = I_3 d\nu_3$, and the transversality condition: $I_a d\nu_a(X) \neq 0$ for all $X \in \mathfrak{g}$. Any map satisfying these conditions is called a G-moment map. Given a point $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ in $\mathbf{R}^3 \otimes \mathfrak{g}$, denote the level set $\nu^{-1}(\zeta)$ by P. Since the map ν is G-equivariant, level sets are invariant if the group G is Abelian or if the point ς is invariant. Assuming that the level set P is invariant, and the action of G on P is free, then the quotient space $N = P/G$ is a smooth manifold.

Joyce proved that the quotient space $N = P/G$ inherits a natural hypercomplex structure [18]. His construction runs as follows. For each point m in the space P, its tangent space is

$$T_m P = \{t \in T_m M : d\nu_1(t) = d\nu_2(t) = d\nu_3(t) = 0\}.$$

Consider the vector subspace

$$U_m = \{t \in T_mP : I_1 d\nu_1(t) = I_2 d\nu_2(t) = I_3 d\nu_3(t) = 0\}.$$

Due to the transversality condition, this space is transversal to the vectors generated by elements in $\mathfrak{g}$. Due to the Cauchy-Riemann condition, this space is a vector subspace of T_mP with co-dimension $\dim \mathfrak{g}$, and hence it is a vector subspace of T_mM with co-dimension $4 \dim \mathfrak{g}$. The same condition implies that, as a subbundle of $TM_{|P}$, U is closed under I_a. We call the distribution U the hypercomplex distribution of the map ν. Let $\pi : P \to N$ be the quotient map. For any tangent vector v at $\pi(m)$, there exists a unique element $\widetilde{v}$ in U_m such that $d\pi(\widetilde{v}) = v$. The hypercomplex structure on N is defined by

$$I_a v = d\pi(I_a \widetilde{v}), \quad \text{i.e.} \quad \widetilde{I_a v} = I_a \widetilde{v}. \tag{31}$$

Theorem 6. *Let $(M, \mathcal{I}, g)$ be a HKT-manifold. Suppose that G is a compact group of hypercomplex isometries. Suppose that ν is a G-moment map such that along the invariant level set $P = \nu^{-1}(\zeta)$, the hypercomplex distribution U is orthogonal to the Killing vector fields generated by the group G, then the quotient space $N = P/G$ inherits a natural HKT-structure.*

Proof. Under the condition of this theorem, the hypercomplex distribution along the level set P is identical to the orthogonal distribution

$$H_m = \{t \in T_mP : g(t, X) = 0, X \in \mathfrak{g}\}.$$

Now, we define a metric structure h at $T_{\pi(m)}N$ as follows. For $v, w \in T_{\pi(m)}N$,

$$h_{\pi(m)}(v, w) = g_m(\widetilde{v}, \widetilde{w}). \tag{32}$$

It is obvious that this metric on N is hyper-Hermitian. To find the hyper-Kähler connection D on the quotient space N, let v and w be locally defined vector fields on the manifold N. They lift uniquely to G-invariant sections $\widetilde{v}$ and $\widetilde{w}$ of the bundle U. As U is a subbundle of the tangent bundle of P, and P is a submanifold of M, we consider $\widetilde{v}$ as a section of TP and $\widetilde{w}$ as a section of $TM_{|P}$. Restricting the hyper-Kähler connection ∇ onto P, we consider $\nabla_{\widetilde{v}}\widetilde{w}$ as a section of $TM_{|P}$. Recall that there is a direct sum decomposition

$$TM_{|P} = U \oplus \mathfrak{g} \oplus I_1\mathfrak{g} \oplus I_2\mathfrak{g} \oplus I_3\mathfrak{g}. \tag{33}$$

Let θ be the projection from $TM_{|P}$ onto its direct summand U. Since $\mathfrak{g}$ is orthogonal to the distribution U, and U is hypercomplex invariant, θ is an orthogonal projection. Define

$$D_v w := d\pi(\theta(\nabla_{\widetilde{v}}\widetilde{w})). \quad \text{i.e.} \quad \widetilde{D_v w} = \theta(\nabla_{\widetilde{v}}\widetilde{w}). \tag{34}$$

Now we have to prove that it is a HKT-connection.

We claim that the connection D preserves the hypercomplex structure. This claim is equivalent to $D_v(I_a w) = I_a D_v w$. Lifting to U, it is equivalent to $\theta(\nabla_{\widetilde{v}} I_a \widetilde{w}) = I_a \theta(\nabla_{\widetilde{v}}\widetilde{w})$. Since the direct sum decomposition is invariant of the hypercomplex structure, the projection map θ is hypercomplex. Therefore, it commutes with the complex structures. Then the above identity is equivalent to $\theta(\nabla_{\widetilde{v}} I_a \widetilde{w}) = \theta(I_a \nabla_{\widetilde{v}}\widetilde{w})$. This identity holds because ∇ is hypercomplex.

To verify that connection D preserves the Riemannian metric h, let u, v, and w be vector fields on N. The identity $uh(v,w) - h(D_u v, w) - h(v, D_u w) = 0$ is equivalent to the following identity on P: $\widetilde{u}g(\widetilde{v},\widetilde{w}) - g(\theta(\nabla_{\widetilde{u}}\widetilde{v}),\widetilde{w}) - g(\widetilde{v},\theta(\nabla_{\widetilde{u}}\widetilde{w})) = 0$. Since θ is the orthogonal projection along $\mathfrak{g}$, the above identity is equivalent to $\widetilde{u}g(\widetilde{v},\widetilde{w}) - g(\nabla_{\widetilde{u}}\widetilde{v},\widetilde{w}) - g(\widetilde{v},\nabla_{\widetilde{u}}\widetilde{w}) = 0$. This identity on P is satisfied because ∇ is a HKT-connection.

Finally, we have to verify that the torsion of connection D is totally skew-symmetric. By definition and the fact that θ is an orthogonal projection, the torsion of D is $T^D(u,v,w) = g(\nabla_{\widetilde{u}}\widetilde{v},\widetilde{w}) - g(\nabla_{\widetilde{v}}\widetilde{u},\widetilde{w}) - g(\widetilde{[u,v]},\widetilde{w})$. Note that $[\widetilde{u},\widetilde{v}]$ is a vector tangent to P such that $d\pi\circ\theta([\widetilde{u},\widetilde{v}]) = [d\pi(\widetilde{u}), d\pi(\widetilde{v})] = [u,v]$. Therefore, $[\widetilde{u},\widetilde{v}]$ and $\widetilde{[u,v]}$ differ by a vector in $\mathfrak{g}$. Since the Killing vector fields are orthogonal to the hypercomplex distribution, $g(\widetilde{[u,v]},\widetilde{w}) = g([\widetilde{u},\widetilde{v}],\widetilde{w})$. Then we have $T^D(u,v,w) = T^\nabla(\widetilde{u},\widetilde{v},\widetilde{w})$. This is totally skew-symmetric because connection ∇ is the Bismut connection on M. $\square$

Suppose that the group G is one-dimensional. Let X be the Killing vector field generated by G. The hypercomplex distribution U and the horizontal distribution H are identical if and only if the 1-forms $I_1 d\nu_1 = I_2 d\nu_2 = I_3 d\nu_3$ are pointwisely proportional to the 1-form $\iota_X g$ along the level set P, i.e. for any tangent vector Y to P, $I_a d\nu_a(Y) = fg(X,Y)$ or equivalent to $d\nu_a = f\iota_X F_a$. In the next example, we shall make use of this observation.

5.1. Example: HKT-Structure on $\mathcal{V}\left(\mathbf{CP}^2\right) = S^1\times(SU(3)/U(1))$. We construct a HKT-structure on $\mathcal{V}\left(\mathbf{CP}^2\right)$ by a $U(1)$-reduction from a HKT-structure on $\mathbf{H}^3\backslash\{0\}$. Choose a hypercomplex structure on $\mathbf{R}^6 \cong \mathbf{C}^3\oplus\mathbf{C}^3$ by

$$I_1(\chi,\varrho) = (i\chi, -i\varrho), \quad I_2(\chi,\varrho) = (i\varrho, i\chi), \quad I_3(\chi,\varrho) = (-\varrho,\chi). \tag{35}$$

It is apparent that the holomorphic coordinates with these complex structures are $(\chi,\overline{\varrho})$, $(\chi+\varrho, \overline{\chi}-\overline{\varrho})$, and $(\varrho - i\chi, \overline{\varrho} - i\overline{\chi})$ respectively.

As in 4.3, the hyper-Kähler potential for the Euclidean metric g on $(\mathbf{C}^3\oplus\mathbf{C}^3)\backslash\{0\}$ is $\mu = \frac{1}{2}(|\chi|^2 + |\varrho|^2)$. We apply Proposition 4 to $f(\mu) = \ln\mu$ to obtain a new HKT-metric

$$\hat{g} = \frac{1}{\mu}g - \frac{1}{\mu^2}(d\mu\otimes d\mu + I_1 d\mu\otimes I_1 d\mu + I_2 d\mu\otimes I_2 d\mu + I_3 d\mu\otimes I_3 d\mu). \tag{36}$$

Define a hypercomplex moment map $\nu = (\nu_1,\nu_2,\nu_3)$ by

$$\nu_1(\chi,\varrho) = |\chi|^2 - |\varrho|^2, \quad (\nu_2 + i\nu_3)(\chi,\varrho) = 2\langle\chi,\varrho\rangle, \tag{37}$$

where $\langle,\rangle$ is a Hermitian inner product on $\mathbf{C}^3$. Let $\Gamma \cong \mathrm{U}(1)$ be the one-parameter group acting on $(\mathbf{C}^3\oplus\mathbf{C}^3)\backslash\{0\}$ defined by

$$(t;(\chi,\varrho)) \mapsto (e^{it}\chi, e^{it}\varrho). \tag{38}$$

Let $\langle r\rangle$ be the integer group generated by a real number between 0 and 1. It acts on $(\mathbf{C}^3\oplus\mathbf{C}^3)\backslash\{0\}$ by

$$(n;(\chi,\varrho)) \mapsto (r^n\chi, r^n\varrho). \tag{39}$$

Both Γ and $\langle r\rangle$ are groups of hypercomplex automorphisms leaving the zero level set of ν invariant. Then the quotient space $\nu^{-1}(0)/\Gamma$ is a hypercomplex reduction. The discrete quotient space $\mathcal{V} = \nu^{-1}(0)/\Gamma \times \langle r \rangle$ is a compact hypercomplex manifold. From the homogeneity of the metric $\hat{g}$, we see that both Γ and the discrete group $\langle r \rangle$ are group of isometries for the metric $\hat{g}$. Therefore, the quotient space $\mathcal{V}$ inherits a hyper-Hermitian metric.

On $(\mathbf{C}^3 \oplus \mathbf{C}^3)\backslash\{0\}$, the real vector field generated by the group Γ is

$$X = i\chi\frac{\partial}{\partial\chi} - i\overline{\chi}\frac{\partial}{\partial\overline{\chi}} - i\overline{\varrho}\frac{\partial}{\partial\overline{\varrho}} + i\varrho\frac{\partial}{\partial\varrho}.$$

Let $\hat{F}_a$ be the Kähler form for the HKT-metric $\hat{g}$. We check that $d\nu_a = -2\mu\iota_X\hat{F}_a$. Therefore, Theorem 6 implies that the quotient space $\mathcal{V}$ inherits a HKT-structure.

Note that if (χ, ϱ) is a point in the zero level set, then it represents a pair of orthogonal vectors. Therefore, the triple $(\frac{\chi}{|\chi|}, \frac{\varrho}{|\varrho|}, \frac{\overline{\chi}}{|\chi|} \times \frac{\overline{\varrho}}{|\varrho|})$ forms an element in the matrix group $\mathrm{SU}(3)$. The action of Γ induces an action on $\mathrm{U}(3)$ by the left multiplication of $\mathrm{Diag}(e^{it}, e^{it}, e^{-2it})$. Denote the Γ-coset of $(\frac{\chi}{|\chi|}, \frac{\varrho}{|\varrho|}, \frac{\overline{\chi}}{|\chi|} \times \frac{\overline{\varrho}}{|\varrho|})$ by $[\frac{\chi}{|\chi|}, \frac{\varrho}{|\varrho|}, \frac{\overline{\chi}}{|\chi|} \times \frac{\overline{\varrho}}{|\varrho|}]$. The quotient space $\mathcal{V}$ is isomorphic to the product space $S^1 \times \mathrm{SU}(3)/\,\mathrm{U}(1)$. The quotient map is

$$(\chi, \varrho) \mapsto \left(\exp\left(2\pi i\frac{\ln|\chi|}{\ln r}\right), \quad \left[\frac{\chi}{|\chi|}, \frac{\varrho}{|\varrho|}, \frac{\overline{\chi}}{|\chi|} \times \frac{\overline{\varrho}}{|\varrho|}\right]\right).$$

Remark. A fundamental question on HKT-structures remains open. Does every hypercomplex manifold admit a metric such that it is a HKT-structure?

Acknowledgements. We thank G. W. Gibbons for introducing the topic in this paper to us and P. Gauduchon and V. Apostolov for useful conversations. The second named author thanks J.-P. Bourguignon for providing an excellent research environment at the Institut des Hautes Études Scientifiques. The first named author thanks the “Abdus Salam International Center for Theoretical Physics” where the final part of this work was done.

References

1. Barberis, M.L., Dotti Miatello, I.G. and Miatello, R.J.: On certain locally homogeneous Clifford manifolds. Ann. Global Anal. Geom. **13**, 289–301 (1995)
2. Besse, A.: *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge **10**, New York: Springer-Verlag, 1987
3. Bismut, J.-M.: A local index theorem for non-Kähler manifolds. Math. Ann. **284**, 681–699 (1989)
4. Boyer, C.: A note on hyperhermitian four-manifolds. Proc. Amer. Math. Soc. **102**, 157–164 (1988)
5. Boyer, C., Galicki, K. and Mann, B.: Hypercomplex structures on Stiefel manifolds. Ann. Global Anal. Geom. **14**, 81–105 (1996)
6. Curtright, and D. Freedman, Z.: Nonlinear σ-models with extended supersymmetry in four dimensions. Phys. Lett. B **90**, 71 (1980)
7. Capria, M.M. and Salamon, S.M.: Yang–Mills fields on quaternionic spaces. Nonlinearity **1**, 517–530 (1988)
8. Dotti. I.G. and Fino, A.: Abelian hypercomplex 8-dimensional nilmanifolds. Preprint
9. Eells, J. and Salamon, S.M.: Twistorial construction of harmonic maps of surfaces into four manifolds. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **12**, 589–640 (1985)
10. Galicki, K. and Poon, Y.S.: Duality and Yang–Mills fields on quaternionic Kähler manifolds. J. Math. Phys. **32**, 1529–1543 (1991)
11. Gates, S.J., Hull, C.M. and Roček, M.: Twisted multiplets and new supersymmetric nonlinear sigma models. Nucl. Phys. B **248**, 157–186 (1984)
12. Gauduchon, P.: Hermitian connections and Dirac operators. Bollettino U.M.I. B **11**, 257–288 (1997)
13. Gauduchon, P. and Tod, K.P.: Hyper-Hermitian metrics with symmetry. J. Geom. Phys. **25**, 291–304 (1998)
14. Gibbons, G.W., Papadopoulos, G. and Stelle, K.S.: HKT and OKT geometries on soliton black hole moduli spaces. Nucl. Phys. B **508**, 623–658 (1997)

15. Hitchin, N.J., Karlhede, A., Lindström, U., and Roček, M.: Hyper-Kähler metrics and supersymmetry. Commun. Math. Phys. **108**, 535–589 (1987)
16. Howe, P.S. and Papadopoulos, G.: Holonomy Groups and W-symmetries. Commun. Math. Phys. **151**, 467–480 (1993)
17. Howe, P.S. and Papadopoulos, G.: Twistor spaces for hyper-Kähler manifolds with torsion. Phys. Lett. B **379**, 80–86 (1996)
18. Joyce, D.: The hypercomplex quotient and quaternionic quotient. Math. Ann. **290**, 323–340 (1991)
19. Joyce, D.: Compact hypercomplex and quaternionic manifolds. J. Differential Geom. **35**, 743–761 (1992)
20. Opfermann, A. and Papadopoulos, G.: Homogeneous HKT and QKT manifolds. Preprint, math-ph/9807026
21. Pedersen, H. and Poon, Y.S.: Deformations of hypercomplex structures. J. reine angew. Math. **499**, 81–99 (1998)
22. Pedersen, H. and Poon, Y.S.: Inhomogeneous hypercomplex structures on homogeneous manifolds. J. reine angew. Math. **516**, 159–181 (1999)
23. Pedersen, H., Poon, Y.S. and Swann, A.F.: Hypercomplex structures associated to quaternionic manifolds. Differential Geom. Appl. **9**(3), 273–292 (1998)
24. Salamon, S.M.: Differential geometry of quaternionic manifolds. Ann. scient. Éc. Norm. Sup. 4^e, **19**, 31–55 (1986)
25. Salamon, S.M.: Complex structures on nilpotent Lie algebras. Preprint math.DG/ 9808025
26. Spindel, Ph., Sevrin, A., Troost, W. and Van Proeyen, A.: Extended supersymmetric σ-models on group manifolds. Nucl. Phys. B **308**, 662–698 (1988)
27. Swann, A.F.: HyperKähler and quaternionic Kähler geometry. Math. Ann. **289**, 421–450 (1991)
28. Zumino, B.: Supersymmetry and Kähler manifolds. Phys. Lett. B **87**, 203–206 (1979)

Communicated by R. H. Dijkgraaf