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TWISTORIAL CONSTRUCTION OF QUATERNIONIC MANIFOLDS

Henrik Pedersen and Y. Sun Poon

In four dimensional geometry it is a well-known result that the twistor space of a self-dual manifold has a naturally defined integrable complex structure [1]. Conversely, a self-dual conformal structure is uniquely determined by a given twistor space. When there is an Einstein metric in the self-dual conformal class, the twistor space is a uniquely determined complex contact manifold. This one-to-one correspondence between self-dual conformal structure and twistor space is generally known as the Penrose correspondence or twistor correspondence.

As it was shown in [7,8], a high dimensional generalization of a self-dual (Einstein) manifold is a quaternionic (Kähler) manifold. Salamon showed that the twistor space of a quaternionic manifold has integrable complex structure and that the twistor space of a quaternionic Kähler manifold is a complex contact manifold. The construction is the following:

By definition, a *quaternionic manifold* is a real $4n$-dimensional manifold ($n \geq 2$) with a G-structure admitting a torsion-free connection, where $G = \mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1)$. When the manifold is *quaternionic Kähler*, its holonomy can be reduced to $\mathrm{Sp}(n)\mathrm{Sp}(1)$. As G acts on the unit 2-sphere in the space of imaginary quaternions through the adjoint representation of $\mathrm{Sp}(1)$, there is always an associated S^2-bundle over the manifold. The total space of this fibre bundle is called the (associated) twistor space. With the given geometrical assumption, the twistor space has a naturally defined complex structure. The fibres are holomorphic rational curves with normal bundle $\underline{C}^{2n} \otimes L$, where $\underline{C}^{2n}$ is the trivial rank $2n$ bundle and L is the hyperplane bundle on a rational curve.

To complete this generalization of the twistor correspondence, we shall prove that a twistor space uniquely determines a quaternionic structure. Moreover, we shall use this result to produce new examples of quaternionic manifolds. However, we shall only be able to assert that the metric is pseudo-Riemannian with holonomy in $\mathrm{Sp}(k, n-k)\mathrm{Sp}(1)$ ($n \geq 2$), i.e. we may get a *pseudo quaternionic*

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To complete this generalization of the twistor correspondence, we shall prove that a twistor space uniquely determines a quaternionic structure. Moreover, we shall use this result to produce new examples of quaternionic manifolds. However, we shall only be able to assert that the metric is pseudo–Riemannian with holonomy in $\mathrm{Sp}(k,n-k)\mathrm{Sp}(1)$ ($n \geq 2$), i.e. we may get a *pseudo quaternionic*

Kähler manifold. The precise statement of our observation is the following (c.f. [2,3]):

Theorem *Let Z be a complex manifold of $2n+1$ dimensions such that*
(i) *there is an antiholomorphic involution τ without fixed points; (this is called a real structure),*
(ii) *Z is fibered by τ-invariant nonsingular rational curves; (these curves are called real twistor lines),*
(iii) *the normal bundle of a real twistor line is $\underline{C}^{2n} \otimes L$.*

Then Z is isomorphic to the twistor space of a quaternionic manifold of $4n$ (real) dimensions.

If, in addition, Z has holomorphic contact structure θ compatible with the real structure, then Z is isomorphic, as a contact manifold, to the twistor space of a pseudo quaternionic Kähler manifold.

We shall prove the first assertion in §1. The proof of the second assertion in §2 is more constructive than the one in §1. Standard examples as well as new examples will be in §3. See also [5] which contains an independent account of the inverse construction for Quaternionic Kähler Manifolds.

1

Since, by assumption, the normal bundle of a real twistor line in Z is $\underline{C}^{2n} \otimes L$, Kodaira's deformation theory [4] ensures that the real twistor lines belong to a complete n-dimensional family of curves in Z. The parameter space is a complex manifold M such that the tangent space T_xM at a point x in M is isomorphic to the space of holomorphic sections of the normal bundle of the twistor line P_x in Z. When P_x is locally defined as $(0,0,\ldots,z)$ and $t \to x(t)$ is a curve on M with $x(0)=x$, then for t sufficiently close to zero, the curve $P_{x(t)}$ in Z with parameter $x(t)$ is given by $(\phi^i(z,t),z)$, $i=1,\ldots,2n$ for some holomorphic functions ϕ^i. Then the isomorphism from T_xM is given by

$$(*) \qquad \frac{\partial}{\partial t} \longrightarrow \left(\frac{\partial \phi^i}{\partial t}_{|t=0}\right) \qquad i=1,\ldots 2n \; .$$

Let X be the parameter space of τ-invariant twistor lines, then X is a smooth $4n$-dimensional manifold. When x is a real point in M, T_xX is a real subspace of T_xM. By assumption, there is fibration p from Z onto X. We want to construct a quaternionic structure on X.

Since any real twistor line has normal bundle $\underline{C}^{2n} \otimes L$, when a point in M is sufficiently close to X, the corresponding twistor line has the same normal bundle. As we are simply interested in X and hence a neighborhood of X in M, we assume that the above assertion on the normal bundle is true all over M.

Therefore, for any x in M,

$$T_xM \simeq H^0(P_x, \underline{C}^{2n} \otimes L) \simeq H^0(P_x, \underline{C}^{2n}) \otimes H^0(P_x, L) .$$

Let $E_x = H^0(P_x, \underline{C}^{2n})$, $H_x = H^0(P_x, L)$. When x runs through a neighborhood in M, we have the locally defined vector bundles E and H with rank $2n$ and 2 respectively.

When x is a real point, the corresponding twistor line P_x is τ–invariant, so is its normal bundle. But the only fixed point free antiholomorphic involution on $\mathbb{CP}^1$ is the antipodal map. Then the antipodal map induces a unique (up to sign) quaternionic structures j_E and j on E_x and H_x respectively such that, on $E_x \otimes H_x$,

$$\tau_*(e \otimes h) = j_E(e) \otimes j(h) .$$

These structures identify the vector spaces to quaternionic right modules:

$$E_x \simeq \mathbb{H}^n , \quad H_x \simeq \mathbb{H} .$$

Then the tangent space of M is $\mathbb{H}^n \otimes \mathbb{H}$. This identification shows that the principal bundle of frames over the τ–invariant manifold X can be reduced to a G–structure where $G = \mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1)$. In other words [8], X is *almost quaternionic*. We are going to use the integrability of the complex structure on Z to show that the obstruction to the existence of a torsion–free G–connection vanishes.[1]

Let $P(H)$ be the total space of the fibre bundle over X with fibre $\mathbb{P}(H^0(P_x, L))$. Since

$$H^0(P_x, L) \simeq H_x$$

$P(H)$ is the twistor space of the almost quaternionic manifold X. Fixing any connection on H, we can define a tautological almost complex structure on $P(H)$ as follows [1,7].

At each point $\underline{h}$ in $P(H)$, the connection defines a splitting of the complexified cotangent space into horizontal and vertical parts,

$$(T^*_{\underline{h}}P)^c = (T^*_{\underline{h}}P_x)^c \otimes (\pi^* T^*_x X)^c$$

where π is the projection from $P(H)$ onto X and $x = \pi(\underline{h})$. The vertical $(0,1)$–form is chosen to be the standard one on the Riemann sphere P_x. The horizontal $(0,1)$–forms at $\underline{h}$ are chosen to be the $2n$–dimensional subspace in

$$(T^*_x X)^c = T^*_x M = E^*_x \otimes H^*_x$$

of the form $E^*_x \otimes jh$, where h is any vector in H_x representing the point $\underline{h}$ in $P(H_x)$ after we identify H_x with H^*_x via the standard symplectic structure.

[1] We would like to thank S. Salamon for his help on our argument.

On the other hand, when one regards a point z in Z as a divisor on the unique real twistor line P_x through z, one has a smooth bundle map ϕ from Z onto $P(H)$. This is a diffeomorphism because it is a bundle map and it is a holomorphic isomorphism on each fibre. We claim that ϕ is holomorphic with respect to the given almost complex structure on Z and the tautological complex structure on $P(H)$. It will be true if ϕ pulls back $(0,1)$–forms on $P(H)$ to $(0,1)$–forms on Z. By construction, this is obviously the case when the $(0,1)$–form on $P(H)$ is vertical. On the other han, all horizontal $(0,1)$–forms at $\phi(z)$, has the form $\pi^*(e^* \otimes jh)$. Since $p = \pi \circ \phi$, then

$$\phi^* \pi^*(e^* \otimes jh) = p^*(e^* \otimes jh) \ .$$

This 1–form at z is of type–$(0,1)$ if it annihilates all $(1,0)$–vectors. Therefore, the claim wiil be proved if one can see that $(dp)_z(v)$ is annihilated by $e^* \otimes jh$ whenever v is a holomorphic tangent vector to Z at z. Note that $(dp)_z$ is the complex linear extension of the differential of the fibre projection from Z onto X, $(dp)_z$ annihilates all vertical vector fields. On the other hand, as the normal bundle N of P_x is $\underline{C}^{2n} \otimes L$ and the holomorphic tangent bundle of P_x is L^2, the exact sequence on P_x

$$0 \longrightarrow TP_x \longrightarrow TZ_{|P_x} \longrightarrow N \longrightarrow 0$$

splits as $TZ_{|P_x} \simeq TP_x \otimes N$.

Therefore we consider the vector $(dp)_z(v)$ where $v \in N_z$. But we have the following exact sequence of vector spaces:

$$0 \longrightarrow H^0(P_x, \mathcal{I}_z(N)) \longrightarrow H^0(P_x, N) \xrightarrow{\epsilon} N_z \longrightarrow 0 \ ,$$

where ϵ is the evaluation of a section at z and $H^0(P_x, \mathcal{I}_z(N))$ is the set of all global sections of the normal bundle on P_x that vanishes at z. Now from the Kodaira's identification, one sees that $\epsilon(dp)_z$ is an automorphism of N_z, therefore, $(dp)_z$ defines a splitting:

$$H^0(P_x, N) \simeq H^0(P_x, \mathcal{I}_z(N)) \oplus N_z \ ,$$

i.e.

$$\begin{aligned} (T_xX)^c &= T_xM \\ &\simeq H^0(P_x, N) \\ &\simeq H^0(P_x, \mathcal{I}_z(N)) \oplus N_z \ . \end{aligned}$$

As the real structure τ on $(T_xX)^c$ sends a section of the normal bundle that vanishes at z to one that vanishes at $\tau(z)$, while any section of N that vanishes at two points is identically zero, the above splitting of T_xM determined by dp identifies N_x to $H^0(P_x, \mathcal{I}_z(N))$. In other words, any $(0,1)$–vector at z is mapped by $(dp)_z$ to be a section of N that vanishes at $\tau(z)$. As an element in $E_x \otimes H_x$, it has the form $e' \otimes jh$. But then

$$(e^* \otimes jh) \cdot (e' \otimes jh) = 0$$

because we use a symplectic form to identify H^* and H. Now we can conclude that ϕ pulls back any $(0,1)$–form on $P(H)$ to $(0,1)$–form on Z. As a consequence, the tautological complex structure on $P(H)$ is integrable.

Now, Salamon's tensor C_0 of obstruction to the existence of a torsion–free G–connection can be identified with the composition [7]:

$$B^2 \subset \Lambda^2 T^*X \xrightarrow{d} \Lambda^3 T^*X \xrightarrow{\text{projec.}} A^3$$

where B^2 is the 2–forms on X which are of type $(1,1)$ with respect to all complex structures and A^3 is the 3–forms of type $(0,3)$ with respect to some complex structure. Since the tautological complex structure on $P(H)$ is integrable, the exterior derivative of any $(1,1)$–form has no $(0,3)$–component. Then, if $\alpha \in B^2$, we have $\pi^*\alpha \in \Lambda^{1,1}P(H)$ and $\pi^*(d\alpha) = d(\pi^*\alpha) \in \Lambda^{2,1}P(H) \oplus \Lambda^{1,2}P(H)$. Thus, the obstruction to the existence of a torsion–free G–connection vanishes.

Caution: One should not conclude from this proof that a complex 3–dimensional twistor space determines a self–dual conformal class on a 4–fold. In this case, our conclusion is simply that there is a torsion–free connection in a $GL(4,\mathbb{R})$–structure. For a proof on the four dimensional case, one may refer to Chap 13 in [2].

2

In this section we assume that Z has an additional structure, namely the contact structure θ.

By definition, θ is a holomorphic section of $\Omega \otimes K^{-1(n+1)}$, i.e. a 1–form with values in the bundle $K^{-1/(n+1)}$, which is a line bundle such that its $(n+1)$-th power is the anticanonical bundle. Moreover, $\theta \wedge (d\theta)^n$ is a nondegenerate holomorphic $2n+1$–form. Considering θ as a bundle map from TZ onto $K^{-1(n+1)}$, the kernel of this map, D, is a holomorphic bundle on Z such that the following sequence is exact:

$$0 \longrightarrow D \longrightarrow TZ \longrightarrow K^{-1/(n+1)} \longrightarrow 0\ .$$

It is known that $K^{-1/(n+1)}$ is isomorphic to TP_x for each real twistor lines P_x in Z [7], so the restriction of D onto real twistor lines is isomorphic to its normal bundle.

As θ is a section of $\Omega \otimes K^{-1/(n+1)}$, $d\theta$ is a well–defined section of $\Lambda^2 D^* \otimes K^{-1/(n+1)}$. While $D_{|P_x} \simeq N \simeq \underline{C}^{2n} \otimes L$ and $K^{-1/(n+1)} \simeq TP_x \simeq L^2$, the restriction of $d\theta$ onto P_x is a 2–form on the vector space

$$H^0(P_x, D \otimes L^{-1}) \simeq E_x\ .$$

Since θ is a contact structure, this 2–form on E_x is nondegenerate. It defines a symplectic structure on the bundle E, denoted by ω^E.

Although the bundle H always has a symplectic form, θ can help to fix our choice. In fact, the Wronskian, W, is a nondegenerate 2–form on H_x. On the other hand, the composition of the inclusion i, from TP_x into $TZ_{|P_x}$ with θ is an isomorphism from TP_x onto L^2. When this map is considered as an automorphism of L^2, it is simply a nonzero constant multiplication. Therefore, we scale the Wronskian and define

$$\omega^H = (\theta \circ i)^{-1} W \ .$$

Then

$$g_c = \omega^E \otimes \omega^H$$

is a nondegenerate complex bilinear form on TM.

Since θ is required to be compatible with the real structure $\tau_* = j_E \otimes j$, we may define, for some k, Sp(1) and Sp($k, n-k$) structures by the Hermitian metrics $g^H(\cdot,\cdot) = \omega^H(j\cdot,\cdot)$ and $g^E(\cdot,\cdot) = \omega^E(j_E\cdot,\cdot)$ on the bundle H and E respectively. Then g is a pseudo Riemannian metric on the τ–invariant space X in M. The form of the metric implies that the principal bundle of frames can be reduced to a Sp(n)Sp(1)–bundle. We have to prove that the Levi–Civita connection of this metric is also reduced. Our method is to construct connections ∇^E and ∇^H respectively from the given structures and then prove that $\nabla^E \otimes \nabla^H$ is the Levi–Civita connection.

Let π be the parameter space of all twistor lines through a given point z in Z. As

$$E_x \simeq H^0(P_x, D \otimes L^{-1}) \simeq H^0(P_x, \underline{C}^{2n}) \ ,$$

the evaluation of a point in E_x at the point z in P_x is a trivialization of $E_{|\pi}$. Then on $E_{|\pi}$, we have the flat connection. To be precise, when v is a tangent vector of π at x and s is a section of $E_{|\pi}$, if ϕ is the evaluation, then

$$\phi(\nabla_v s) = d_v(\phi s) \ . \tag{1}$$

When T_xM is identified to $E_x \otimes H_x$, a vector in T_xM is tangent to π if and only if it is a simple tensor product. It will be called a *simple* tangent vector. The collection of all these simple directions in T_xM is the algebraic submanifold $P(E_x) \times P(H_x)$ embedded in $P(T_xM)$ by the tensor product map. Since a connection is defined on $E_{|\pi}$ for all π, the connection matrix at x is a $2n \times 2n$ matrix of holomorphic functions of the simple tangent vectors at x. As the connection depends on a given direction linearly, the entries of the connection matrix can be considered as a section of the restriction of the hyperplane bundle from $\mathsf{P}(T_xM)$ onto $\mathsf{P}(E_x) \times \mathsf{P}(H_x)$. One can show that any such section over $V = P(E_x) \times P(H_x)$ can uniquely be extended to a hyperplane section over $P(T_xM)$ as follows: As V is embedded into $\mathsf{P}^{4n-1} = P(T_xM)$ by the Segre embedding, sections of the hyperplane bundle is naturally restricted to be sections of the restricted bundles. This linear map is injective because the image of the Segre embedding is not contained in any linear subspace of $\mathbb{P}^{4n-1}$. Therefore, sections of the restriction of the hyperplane bundle are uniquely extended if and only if the dimension of the restricted bundle is equal to the dimension of the space of sections of the hyperplane bundle, i.e. $4n$. Note that any line bundle on V is uniquely determined by its bidegree. In particular, the restriction of the hyperplane has bidegree $(1,1)$. Let D be a divisor obtained by fixing a point in the second factor of V. Then D has bidegree $(0,1)$. Then on V, one has the exact sequence:

$$0 \longrightarrow \mathcal{O}(1,0) \longrightarrow \mathcal{O}(1,1) \longrightarrow \mathcal{O}_D(1,1) \longrightarrow 0 .$$

Since $D \cdot (1,1) = 1$, the restriction of $\mathcal{O}(1,1)$ to D is the hyperplane bundle on D which is a copy of projective space. Then the above exact sequence induces:

$$0 \longrightarrow H^0(V, \mathcal{O}(1,0)) \longrightarrow H^0(V, \mathcal{O}(1,1)) \longrightarrow \mathbb{C}^{2n} \longrightarrow H^1(V, \mathcal{O}(1,0)) \longrightarrow 0 .$$

We shall prove, by induction, that $h^0(V, \mathcal{O}(1,0)) = 2n$, and $h^1(V, \mathcal{O}(1,0)) = 0$. Let $V_1 = \mathbb{P}^{2n-2} \times \mathbb{P}^1$. It is a divisor of V with bidegree $(1,0)$. Therefore,

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1,0) \longrightarrow \mathcal{O}_{V_1}(1,0) \longrightarrow 0 .$$

Using the fact that the Hodge numbers of V satisfy $h^{0,0} = 1$, $h^{0,1} = 0$, $h^{0,2} = 0$ we have

$$0 \longrightarrow \mathbb{C} \longrightarrow H^0(V, \mathcal{O}(1,0)) \longrightarrow H^0(V_1, \mathcal{O}_{V_1}(1,0)) \longrightarrow 0 ,$$

and

$$0 \longrightarrow H^1(V, \mathcal{O}(1,0)) \longrightarrow H^1(V_1, \mathcal{O}(1,0)) \longrightarrow 0 .$$

However, the restriction of the $(1,0)$ bundle from $\mathbb{P}^{2n-1} \times \mathbb{P}^1$ to $\mathbb{P}^{2n-1} \times \mathbb{P}^1$ is exactly the $(1,0)$ bundle. Therefore, the above expression gives, inductively,

$$h^0(\mathbb{P}^{2n-1} \times \mathbb{P}^1, \mathcal{O}(1,0)) = 2n - 2 + h^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,0)) = 2n$$

and

$$h^1(\mathbb{P}^{2n-1} \times \mathbb{P}^1, \mathcal{O}(1,0)) = h^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,0)) = 0 .$$

Therefore, the exact sequence

$$0 \to H^0(V, \mathcal{O}(1,0)) \to H^0(V, \mathcal{O}(1,1)) \to H^0(\mathbb{P}^{2n-1}, \mathcal{O}(1)) \to H^1(V, \mathcal{O}(1,0)) \to 0$$

implies that

$$h^0(V, \mathcal{O}(1,1)) = 4n .$$

This finishes the proof that any section can be extended uniquely.

Therefore, the entries of the connection matrix is actually linear in all directions. Then the connection is uniquely extended onto M. This connection on E is denoted by ∇^E.

We claim that $\nabla^E \omega^E = 0$. It suffices to see that $\nabla^E_v \omega^E = 0$ where v is simple at x. When v is simple, v is tangent to a subspace π containing x and parametrice all twistor lines through a given point z. But for any y in π, ω^E is the restriction of the global section $d\theta$ on P_y. As $d\theta$ is a section of the trivial bundle $D \otimes L^{-1}$ on P_y, its value is determined by the value of $d\theta$ at z. It means that ω^E is independent of the choice of the point in π and hence $\nabla^E \omega^E = 0$.

Using the identification of Z and $P(H)$ in the last section, we can define a connection on H simply by the distribution D. This distribution defines a complex structure on $P(H)$ and $P(H)$ is biholomorphic to Z as was proved in §2. To see the parallel transport of this ∇^H on H, we take a point x in M and a simple vector v at x. Then v is a section of the normal bundle of P_x vanishing at a point z. Since the distribution D on P_x is isomorphic to the normal bundle, we consider this section as a normal vector field $\underline{v}$ along P_x.

On the other hand, let π be the space of all twistor lines through z, $\alpha(t)$ a path in π originated at x with direction v. If s is a local section of H defined in a neighborhood of x, then s_x is a section of L over P_x. Suppose that the zero of s_x on P_x is a point z' not equal to z, then $\underline{v}(z')$ is a normal tangent vector of P_x at z'. The integral curve of $\underline{v}(z')$ will meet $P_{\alpha(t)}$ when t is small. When this point is considered as a divisor on $P_{\alpha(t)}$, we have a section of L over $P_{\alpha(t)}$ with undetermined scale. The parallel transport from s_x to $s'_{\alpha(t)}$ in the direction v is chosen so that $s_x(z) = s'_{\alpha(t)}(z)$. This equality makes sense because P_x and $P_{\alpha(t)}$ intersect at z and the bundle $K^{-1/(n+1)}$ is defined in a neighborhood of the twistor line P_x. The above equality is then between elements in $K_z^{-1/(n+1)}$. To realize that this connection on H is the one defined by the distribution D, we only have to observe that $\underline{v}$ is a horizontal vector field with respect to both θ and the connection that we have just described. However, in this description of ∇^H, it is easy to see that $\nabla^H \omega^H = 0$ because differentiations normal to the fibres commute with differentiation along the fibres and therefore the connection respects the Wronskian. Since

$$\nabla^E \omega^E = 0 \ , \quad \nabla^H \omega^H = 0$$

and

$$g_c = \omega^E \otimes \omega^H \ ,$$

$\nabla = \nabla^E \otimes \nabla^H$ is a metric connection on TX with respect to the metric g_c.

In the above discussion, a section s of H is considered as a function on the twistor space $P(H)$, homogeneous of degree 1 in fibre coordinate. Then the

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In the above discussion, a section s of H is considered as a function on the twistor space $P(H)$, homogeneous of degree 1 in fibre coordinate. Then the covariant derivative of s with respect to the tangent vector v is simply the directional derivative of the function with respect to the normal vector field $\underline{v}$ on the twistor space, i.e.

(2) $$\nabla^H_v s = \underline{v}(s)\ .$$

Now let $v_k = e_k \otimes h_k$, $k = 1, 2$, be two nonzero simple vectors at x. Let z_k be the zero of h_k on P_x, then

$$\begin{aligned}
&(\nabla_{v_0} v_1 - \nabla_{v_1} v_0)(z_0) \\
&\quad = [\nabla_{v_0}(e_1 \times h_1) - \nabla_{v_1}(e_0 \otimes h_0)](z_0) \\
&\quad = (\nabla^E_{v_0} e_1)(z_0) \otimes h_1(z_0) + e_1(z_0)(\nabla^H_{v_0} h_1)(z_0) - e_0(z_0)(\nabla^H_{v_1} h_0)(z_0) \\
&\quad = \underline{v}_0(e_1(z_0))h_1(z_0) + e_1(z_0)\underline{v}_0(h_1(z_0)) - e_0(z_0)\underline{v}_1(h_0(z_0)) \\
&\quad = \underline{v}_0(e_1(z_0))h_1(z_0) + e_1(z_0)\underline{v}_0(h_1(z_0)) - e_0(z_0)\underline{v}_1(h_0(z_0)) - \underline{v}_1(e_0(z_0))h_0(z_0) \\
&\quad = \underline{v}_0(\underline{v}_1(z_0)) - \underline{v}_1(\underline{v}_0(z_0)) \\
&\quad = [\underline{v}_0, \underline{v}_1](z_0)\ .
\end{aligned}$$

Therefore, the section of N:

$$\nabla_{v_0} v_1 - \nabla_{v_1} v_0 - [v_0, v_1]$$

vanishes at z_0 and by symmetry at z_1. But a section on N vanishes at two distinct points only if it is identically zero. Therefore, the torsion of the connection ∇ vanishes. As a conclusion, $\nabla = \nabla^E \otimes \nabla^H$ is the Levi–Civita connection of g. By construction, it is a $\mathrm{Sp}(k, n-k)\mathrm{Sp}(1)$ connection. Therefore, g is a pseudo quaternionic Kähler metric on X. Moreover, since the horizontal distribution of θ coincides with the one given by the connection on H induced from the metric on X, our contact manifold Z is indeed isomorphic to the associated twistor space of Z.

Remark: The last part of our argument is esentially the same as in dimension four [9].

3

Example 1. Let $\pi\colon \mathbb{CP}^{2n+1} \longrightarrow \mathbb{HP}^n$ be given by

$$\pi(z_0, \ldots, z_{2n+1}) = (z_0 + z_1 j, \ldots, z_{2n} + z_{2n+1} j)$$

and

$$\tau(z_0, \ldots, z_{2n+1}) = (-\bar{z}_1, \bar{z}_0, \ldots, -\bar{z}_{2n+1}, \bar{z}_{2n})$$

$$\theta = (z_0\, dz_1 - z_1\, dz_0) + \cdots + (z_{2n}\, dz_{2n+1} - z_{2n+1}\, dz_{2n})\ .$$

Then $\mathbf{CP}^{2n+1}$ is the twistor space associated to the symmetric space $\mathbf{HP}^n$ with real structure τ and contact structure θ. $\mathbf{HP}^n$ is one of those well-known examples of quaternionic Kähler manifolds.

Remark: One may take

$$\theta = (z_0\, dz_1 - z_1\, dz_0 + z_2\, dz_3 - z_3\, dz_2) - (z_4\, dz_5 - z_5\, dz_4 + z_6\, dz_7 - z_7\, dz_6)$$

to get a SU(2,2) metric.

Example 2. Let Z be the flag manifold:

$$Z = \{(z,w) \in \mathbf{CP}^{n+1} \times (\mathbf{CP}^{n+1})^* \mid \sum_{i=0}^{n+1} z_i w_i = 0\} ,$$

and G the Grassmannian of n spaces in $\mathbf{C}^{n+2}$. Define

$$\pi: Z \longrightarrow G \quad \text{by} \quad \pi(z,w) = w \cap z^{\perp} .$$

Here we use a hermitian metric on $\mathbf{C}^{n+2}$ to define the $(n+1)$-dimensional orthogonal complement of the direction z.

$\tau: Z \longrightarrow Z$ is defined as

$$\tau(z,w) = (\bar{\omega}, \bar{z}) .$$

When Z is considered as $\mathbf{P}(T^*\mathbf{CP}^{n+1})$, it has a canonical contact structure:

$$\theta = \mathrm{i} \sum_{j=0}^{n+1} w_j\, dz_j .$$

Then Z is the twistor space of the symmetric space G with real structure τ and contact structure θ. This G is another well-known example of quaternionic Kähler manifold.

Example 3. Let S be a complex manifold of dimension $n+1$ containing a rational curve with normal bundle $L^2 \otimes \underline{C}^n$. Let Z be the projectivized tangent bundle $\mathbf{P}(TS)$. It is a $2n+1$-dimensional manifold. If we denote by N the normal bundle of the given rational curve in S and $\tilde{N}$ the normal bundle of the canonical lifting of this curve onto Z, then

$$\tilde{N} \simeq J_1(N) ,$$

i.e. the 1-jet bundle. Using the standard exact sequence on this curve:

$$0 \longrightarrow \Omega \otimes N \longrightarrow J_1(N) \longrightarrow N \longrightarrow 0 ,$$

and that $N \simeq L^2 \otimes \underline{C}^n$, $\Omega \simeq L^{-2}$, we see that $\tilde{N} \simeq L \otimes \underline{C}^{2n}$.

Suppose that τ is an antiholomorphic involution without fixed point such that the curve C on S is τ-invariant, then τ is naturally lifted to be a real structure on Z. In this way, we may obtain quaternionic manifolds if only we can find manifolds S as above: Let

$$S = \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1 .$$

The real structure is the antipodal map on each fibre. Let C be any real curve with degree $(1, k_1, k_2, \ldots, k_n)$. In affine coordinate, such a curve can be expressed as

$$t \longrightarrow \left(t, \frac{p_1(t)}{q_1(t)}, \ldots, \frac{p_n(t)}{q_n(t)}\right)$$

where p_j, q_j are polynomials with degree k_j. Then let D_j be the divisor on S defined by the meromorphic function $p_j(t)/q_j(t)$ on the $(j+1)$–factor of $\mathbb{C}P^1$ in S, then C can be considered as the complete intersections of the divisors $D_1, D_2, \ldots, D_n$ and the normal bundle of C in S is isomorphic to $L^{2k_1} \otimes \cdots \otimes L^{2k_n}$. Let $\tilde{S}$ be the locally defined branched covering of the union of the divisors $D_1, D_2, \ldots, D_n$. Then $\tilde{S}$ is an open space containing C and the normal bundle of C in $\tilde{S}$ is reduced to $L^2 \otimes \cdots \otimes L^2 \simeq L^2 \otimes \mathbb{C}^n$. This construction is a generalization of [6].

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