Reduction of HKT-structures

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KT-geometry is the geometry of a Hermitian connection whose torsion is a 3-form. HKT-geometry is the geometry of a hyper-Hermitian connection whose torsion is a 3-form. We identify nontrivial conditions for a reduction theory for these types of geometry. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1487443]

I. INTRODUCTION

Symplectic reduction is a novel method of constructing symplectic manifolds from others that admit a group action of symplectic diffeomorphisms. To describe the main result, let G be a compact group of symplectic diffeomorphisms acting on the symplectic manifold (M,ω) and $\mathfrak{g}$ be the Lie algebra of G. It can be shown that under certain conditions, $N=\nu^{-1}(\zeta)/G$ is also a symplectic manifold, where $\zeta\in\mathfrak{g}^*$ and $\nu:M\to\mathfrak{g}^*$ is the moment map. The manifold N is also denoted with $M//G$. It is remarkable that symplectic reduction can be generalized in various ways. First, it can be shown that if M is a Kähler manifold admitting a G-action of holomorphic isometries, then $M//G$ is also a Kähler manifold. Furthermore, it can be shown that if M is a hyper-Kähler manifold admitting an G-action of triholomorphic isometries, then $M//G=\nu^{-1}(\zeta)$ is also hyper-Kähler, where $\nu=(\nu_1,\nu_2,\nu_3):M\to\mathbf{R}^3\otimes\mathfrak{g}^*$ and $\zeta=(\zeta_1,\zeta_2,\zeta_3)\in\mathbf{R}^3\otimes\mathfrak{g}^*$.[1] In the context of hyper-Kähler reduction there are three moment maps each associated to the three complex structures. One common feature of all symplectic, Kähler and hyper-Kähler reductions is that moment maps exist because the G-action preserves some symplectic form.

More generally it has been shown that if M is a hypercomplex manifold admitting a triholomorphic group action, then $M//G$ is also hypercomplex.[2] The details of this construction will be summarized in Sec. II B. Here it is worth mentioning that in the context of hypercomplex reductions, moment maps do not arise naturally because in the generic case there are no symplectic forms which are preserved by the group action. Instead it is assumed that one can find such functions on M which have the required properties.

In the next section, we assume the existence of a G-moment map on M and study the geometry on the reduced space N. The aim is to prove that the reduction of a KT-space is a KT-space and the reduction of a HKT-space is again a HKT-space. The definition and twistor construction of HKT spaces have been given in Ref. 3. The properties of KT and HKT manifolds have been widely investigated in the literature.[3–5] The result on KT-space in Sec. II is not surprising because a Hermitian structure can easily be found on a reduced space and every Hermitian structure has a unique KT-connection. The existence of HKT-connection on the reduction of a

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HKT-space is less trivial. Examples of HKT-reduction in this regard are given at the end of this paper.

In the third section, we identify nontrivial and sufficient topological or cohomological constraints on either the manifold M or the group G to ensure the existence of a G-moment map on strong KT-manifolds and strong HKT-manifolds. In the absence of symplectic forms, this is a nontrivial result as one usually generates moment map through the Kähler form. In the fourth section, we discuss when a potential function on a HKT-space may descend to a potential function on the reduced HKT-space.

II. EXISTENCE OF HKT-STRUCTURES ON REDUCED SPACES

Assuming the existence of ''moment maps,'' we examine the geometry on the reduced space in the next two sections.

A. KT reduction

Before we explain HKT reduction, it is instructive to consider first the reduction of KT manifolds, i.e., Hermitian manifolds equipped with the hermitian connection whose torsion is a three-from.

Let M be a KT manifold and let G be a compact group of complex isometries on M. Denote the algebra of holomorphic vector fields by $\mathfrak{g}$. Next introduce a G-equivariant map $\nu:M\rightarrow\mathfrak{g}$ satisfying the transversality condition, i.e., $Id\nu(X)\neq 0$ for all $X\in\mathfrak{g}$. We remark that a map ν is equivariant if $\nu(g\cdot x)=\mathrm{Ad}\,g^*(\nu(x))$.

Definition 1: A map ν is called G-moment map if and only if (i) it is equivariant and (ii) it satisfies the transversality condition.

We remark that for simply connected Kähler manifolds the moment map can be constructed using the invariance of Kähler form and complex structure and it satisfies the transversality property. However additional conditions are required in order the moment map to be equivariant.

Next given a point $\zeta\in\mathfrak{g}$, denote the level set $\nu^{-1}(\zeta)$ by P. Since the map ν is G-equivariant, level sets are invariant if the group G is Abelian or if the point ς is invariant. Assuming that the level set P is invariant, and the action of G on P is free, then the quotient space $N=P/G$ is a smooth manifold. Let $\pi:P\rightarrow N$ be the quotient map.

It can be shown that in fact $N=P/G$ is a complex manifold. This construction can be done as follows. For each point m in the space P, its tangent space is

$$T_mP=\{t\in T_mM:d\nu(t)=0\}.$$

Consider the vector subspace

$$\mathcal{U}_m=\{t\in T_mP:Id\nu(t)=0\}.$$

Due to the transversality condition, this space is transversal to the vectors generated by elements in $\mathfrak{g}$. In addition, this space is a vector subspace of T_mP with co-dimension $\dim\mathfrak{g}$, and hence it is a vector subspace of T_mM with co-dimension $2\dim\mathfrak{g}$. The same condition implies that, as a subbundle of $TM_{|P}$, $\mathcal{U}$ is closed under I. Moreover there is a G-invariant splitting,

$$TP=\mathcal{U}\oplus\mathcal{V}, \tag{1}$$

where $\mathcal{V}$ is the tangent space to the orbits of G and it is the bundle of kernels of $d\pi$. We use the terms ''horizontal'' and ''vertical'' for $\mathcal{U}$ and $\mathcal{V}$.

As the projection π is an isomorphism on $\mathcal{U}$, for any tangent vector $\hat{A}$ at $\pi(m)$, there exists a unique element A^u in $\mathcal{U}_m$ such that $d\pi(A^u)=\hat{A}$. We call A^u horizontal lift of $\hat{A}$. The complex structure on N is defined by

$$I\hat{A}=d\pi(IA^u),\quad \text{i.e.,}\quad (I\hat{A})^u=IA^u. \tag{2}$$

Theorem 1: *Let $(M,\mathcal{I},g)$ be a KT-manifold. Suppose that G is a compact group of complex isometries admitting a G-moment map ν. Then the complex reduced space $N=M//G$ inherits a KT structure*.

Proof: To show this, it suffices to find a complex structure I and a Hermitian metric g on N which are induced from M because for every Hermitian structure (I,g), there always exists a unique KT structure on N.[5,6]

To begin, since $\mathcal{U}$ is G-invariant, if X^u is tangent to P at m and is contained in $\mathcal{U}$, then for any element $f\in G$, $dL_f(X^u)$ is tangent to P at $f(m)$ and is contained in $\mathcal{U}$. Using $\pi\circ L_f=\pi$, if X^u is a horizontal lift of $\hat{X}$ to a point m, then $d\pi\circ dL_f(X^u)=d\pi(X^u)=\hat{X}$. Therefore, $dL_f(X^u)$ is the horizontal lift of $\hat{X}$ to $f(m)$.

Since G is also a group of isometries, $g(dL_f(X),dL_f(Y))=g(X,Y)$ for any vectors X and Y tangent to P. Define a metric $\hat{g}$ on N by

$$\hat{g}_{\pi(p)}(\hat{X},\hat{Y})=g_p(X^u,Y^u), \tag{3}$$

where X^u and Y^u are the horizontal lifts of $\hat{X}$ and $\hat{Y}$, respectively. From the analysis above, the metric $\hat{g}$ is independent from the choice of the reference point p of the orbit. Note that the "horizontal" and "vertical" spaces ARE NOT necessarily orthogonal.

To prove that $\hat{g}$ is Hermitian, we note that

$$\begin{aligned} g_{\pi(p)}(I\hat{X},I\hat{Y})&=g_p((I\hat{X})^u,(I\hat{Y})^u)\\ &=g_p(I(\hat{X}^u),I(\hat{Y})^u)=g_p(\hat{X}^u,\hat{Y}^u)=g_{\pi(p)}(\hat{X},\hat{Y}). \end{aligned} \tag{4}$$

Q.E.D.

B. HKT reduction

We shall begin with a description of hypercomplex reduction developed by Joyce.[2] Let G be a compact group of hypercomplex automorphism on M. Denote the algebra of hyper-holomorphic vector fields by $\mathfrak{g}$. Suppose that $\nu=(\nu_1,\nu_2,\nu_3):M\to\mathbf{R}^3\otimes\mathfrak{g}$ is a G-equivariant map satisfying the following two conditions. The Cauchy–Riemann condition, $I_1 d\nu_1=I_2 d\nu_2=I_3 d\nu_3$, and the transversality condition, $I_a d\nu_a(X)\neq 0$ for all $X\in\mathfrak{g}$. In analogy with a similar definition given in the previous section, any map satisfying these conditions is called a G-moment map. Given a point $\zeta=(\zeta_1,\zeta_2,\zeta_3)$ in $\mathbf{R}^3\otimes\mathfrak{g}$, denote the level set $\nu^{-1}(\zeta)$ by P. Assuming that the level set P is invariant, and the action of G on P is free, then the quotient space $N=P/G$ is a smooth manifold.

Joyce proved that the quotient space $N=P/G$ inherits a natural hypercomplex structure.[2] His construction runs as follows. For each point m in the space P, its tangent space is

$$T_mP=\{t\in T_mM: d\nu_1(t)=d\nu_2(t)=d\nu_3(t)=0\}.$$

Consider the vector subspace,

$$\mathcal{U}_m=\{t\in T_mP: I_1 d\nu_1(t)=I_2 d\nu_2(t)=I_3 d\nu_3(t)=0\}. \tag{5}$$

Due to the transversality condition, this space is transversal to the vectors generated by elements in $\mathfrak{g}$. Due to the Cauchy–Riemann condition, this space is a vector subspace of T_mP with co-dimension $\dim\mathfrak{g}$, and hence it is a vector subspace of T_mM with co-dimension $4\dim\mathfrak{g}$.

The same condition implies that, as a subbundle of $TM_{|P}$, $\mathcal{U}$ is closed under I_a. Moreover there is a G-invariant splitting,

$$TP=\mathcal{U}\oplus\mathcal{V}, \tag{6}$$

where $\mathcal{V}$ is the tangent space to the orbits of G and it is the bundle of kernels of $d\pi$. Again, we use the terms ''horizontal'' and ''vertical'' for $\mathcal{U}$ and $\mathcal{V}$ although these two spaces are not necessarily orthogonal. Following techniques and notations of the last section, a hypercomplex structure on N is defined by

$$I_a\hat{A}=d\pi(I_aA^u), \quad \text{i.e.,} \quad (I_aA)^u=I_aA^u. \tag{7}$$

Theorem 2: *Let $(M,\mathcal{I},g)$ be a HKT-manifold. Suppose that G is a compact group of hypercomplex isometries admitting a G-moment map ν. Then hypercomplex reduced space $N=M//G$ inherits a HKT structure.*

Proof: Define hypercomplex structures I_a on $N=P/G$ as in (7). As in the previous section, define a metric $\hat{g}$ on N by

$$g_p(X^u,Y^u)=\hat{g}_{\pi(p)}(\hat{X},\hat{Y}), \tag{8}$$

where X^u and Y^u are the horizontal lifts of $\hat{X}$ and $\hat{Y}$, respectively. This is a hyper-Hermitian metric.

On M, define $F_a(X,Y)=g(I_aX,Y)$ and

$$\omega_1=F_2-iF_3. \tag{9}$$

This is a (0,2)-form with respect to I_1. Since the hyper-Hermitian structure on X admits a HKT-metric, $\overline{\partial}\omega_1=0$. Equivalently, the (0,3)-part of $d\omega_1$ vanishes.

Similarly, we define $\hat{\omega}_1$ on N. By (Ref. 5, Proposition 2), the hyper-Hermitian metric $\hat{g}$ is a HKT-metric if and only if $\overline{\partial}\hat{\omega}_1=0$. In other words, we need to prove that the type (0,3)-part of $d\hat{\omega}_1$ with respect to I_1 vanishes. This is equivalent to

$$\pi^*d\hat{\omega}_1(X^u,Y^u,Z^u)=0 \tag{10}$$

for any vectors X^u,Y^u,Z^u in $\mathcal{U}^{0,1}_{I_1}$. As

$$\pi^*\hat{\omega}_1(Y^u,Z^u)=\omega_1(Y^u,Z^u) \tag{11}$$

and we have the following computation:

$$\begin{aligned} d\pi^*\hat{\omega}_1(X^u,Y^u,Z^u) &= X^u(\pi^*\hat{\omega}_1(Y^u,Z^u))-Y^u(\pi^*\hat{\omega}_1(Z^u,X^u))+Z^u(\pi^*\hat{\omega}_1(X^u,Y^u)) \\ &\quad -\pi^*\hat{\omega}_1([X^u,Y^u],Z^u)-\pi^*\hat{\omega}_1([Y^u,Z^u],X^u)-\pi^*\hat{\omega}_1([Z^u,X^u],Y^u) \\ &= X^u(\omega_1(Y^u,Z^u))-Y^u(\omega_1(Z^u,X^u))+Z^u(\omega_1(X^u,Y^u))-\omega_1([X^u,Y^u]^u,Z^u) \\ &\quad -\omega_1([Y^u,Z^u]^u,X^u)-\omega_1([Z^u,X^u]^u,Y^u)=d\omega_1(X^u,Y^u,Z^u) \\ &\quad +\omega_1([X^u,Y^u]^v,Z^u)+\omega_1([Y^u,Z^u]^v,X^u)+\omega_1([Z^u,X^u]^v,Y^u) \\ &= \omega_1([X^u,Y^u]^v,Z^u)+\omega_1([Y^u,Z^u]^v,X^u)+\omega_1([Z^u,X^u]^v,Y^u). \end{aligned}$$

To complete the proof of this theorem we claim that $[X^u,Y^u]^v=0$. Equivalently, $d_a\nu_a([X^u,Y^u])=0$ for $a=1,2,3$. Since X^u and Y^u are in the kernel of $d_a\nu_a$ for $a=1,2,3$,

$$dd_a\nu_a(X^u,Y^u)=X^u(d_a\nu_a(Y^u))-Y_a(d_a\nu_a(X^u))-d_a\nu_a([X^u,Y^u])=-d_a\nu_a([X^u,Y^u]).$$

As $dd_1\nu_1$ is of type-(1,1) with respect to I_1 and X^u and Y^u are type-(0,1) with respect to I_1, $dd_1\nu_1(X^u,Y^u)=0$. By the Cauchy–Riemann condition $d_1\nu_1=d_2\nu_2=d_3\nu_3$, our claim follows.

Q.E.D.

III. MOMENT MAPS FOR STRONG KT AND HKT SPACES

As we have seen, the construction of new HKT manifolds using HKT reduction requires the existence of a G-moment map satisfying the requirements of Theorem 2. This moment map is not specified within the theory, as it is the case for the hyper-Kähler reduction, but rather its existence is an additional assumption of the construction. However as we shall see in the special case of reduction for strong KT (and HKT) manifolds, under certain assumptions, there is such a moment map which arises naturally. The local construction of a moment map for KT and HKT geometries presented below parallels the construction of an action for two-dimensional (2,0)- and (4,0)-supersymmetric gauged sigma models with the Wess–Zumino term in Ref. 7, respectively. Again, we focus on a reduction theory for strong KT-structure first. The reduction theory for strong HKT-structures follow.

A. Local consideration

Let G a compact group of complex automorphisms on a *strong* KT manifold M. In particular G is a group of isometries on M which leaves in addition the torsion three-form H invariant. To continue we introduce a basis $\{e_a; a=1,\ldots,\dim\mathfrak{g}\}$ in the Lie algebra of $\mathfrak{g}$ and denote the associated vector fields of M with $\{X^a; a=1,\ldots,\dim\mathfrak{g}\}$; denote with $\{e^a; a=1,\ldots,\dim\mathfrak{g}^*\}$ the associated basis in the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. The conditions for invariance of the KT structure can now be written as

$$\mathcal{L}_a g=0, \quad \mathcal{L}_a H=0, \quad \mathcal{L}_a I=0, \tag{12}$$

where $\mathcal{L}_a=\mathcal{L}_{X^a}$; similarly later for the inner derivation we have $i_a=i_{X^a}$.

Using the assumption that M is a strong KT manifold, $dH=0$, the last equation above implies that $di_aH=0$ and so there is a locally defined one-form u_a such that

$$i_aH=du_a.$$

Clearly u_a is uniquely defined up to the addition of a closed one-form.

Next let us denote with $\tilde{X}$ the one-form dual with the vector field X with respect to the KT metric. Using $\mathcal{L}_aI=0$, one can show that the two-form $d(\tilde{X}_a+u_a)$ is type-(1,1) with respect to the complex structure I. Therefore, by the $\bar{\partial}$-Poincare Lemma, there is a locally defined complex-valued function h_a on M such that $(\tilde{X}_a+u_a)^{1,0}=\partial h_a$. Let f_a be the real part of h_a. Define

$$w_a=\tilde{X}_a+u_a-df_a. \tag{13}$$

Then $w_a^{1,0}=i\partial\nu_a$, where ν_a is a constant multiple of the imaginary part of h_a. Therefore, we can write

$$w_a=Id\nu_a. \tag{14}$$

Let $\xi=\xi^a e_a$ be any element in $\mathfrak{g}$. Define a map ν from M to $\mathfrak{g}^*$ by

$$\nu(x)(\xi):=\sum_a \xi^a\nu_a(x). \tag{15}$$

A necessary condition for ν to be well-defined on M is that the class of i_aH in $H^2(M,\mathbf{R})$ should be trivial. If in addition M satisfies the $\partial\bar{\partial}$-lemma, then ν will be well-defined on M.

In the case when the group G is Abelian, the issue of equivariance is absent and hence the map ν so constructed is the moment map. Before we investigate equivariance in general, we consider the issue of nondegeneracy.

Definition 2: A holomorphic Killing vector field X is nondegenerate if $d\nu_X \neq 0$.

Therefore, a holomorphic Killing vector field is nondegenerate if its moment map is nonconstant. The following proposition is useful to determine when a holomorphic Killing vector field is nondegenerate.

Proposition 1: If the length of a holomorphic Killing vector field is nonconstant, then the vector field is nondegenerate.

Proof: Note that $d\nu_X = 0$ if and only if $d\tilde{X} + du = 0$, i.e., $d\tilde{X} + \iota_X H = 0$. It means that for any vector field Y and Z,

$$Y(g(X,Z)) - Z(g(X,Y)) - g(X,[Y,Z]) + H(X,Y,Z) = 0,$$

$$\text{i.e.,} \quad g(\nabla_Y X,Z) + g(X,\nabla_Y Z) - g(\nabla_Z X,Y) - g(X,\nabla_Z Y) - g(X,[Y,Z]) + H(X,Y,Z) = 0,$$

$$\text{or} \quad g(\nabla_Y X,Z) - g(\nabla_Z X,Y) + 2H(X,Y,Z) = 0.$$

On the other hand, since $\mathcal{L}_X g = 0$ and $\nabla g = 0$,

$$\begin{aligned} 0 &= X(g(Y,Z)) - g([X,Y],Z) - g(Y,[X,Z]) \\ &= g(\nabla_X Y,Z) + g(Y,\nabla_X Z) - g([X,Y],Z) - g(Y,[X,Z]) \\ &= g(\nabla_Y X,Z) + g([X,Y],Z) + H(X,Y,Z) + g(Y,\nabla_Z X) \\ &\quad + g(Y,[X,Z]) + H(Y,X,Z) - g([X,Y],Z) - g(Y,[X,Z]) \\ &= g(\nabla_Y X,Z) + g(Y,\nabla_Z X). \end{aligned}$$

Combining the above two identities, we find that for any vector fields Y, Z,

$$g(\nabla_Y X,Z) = -H(X,Y,Z) = -du_X(Y,Z). \tag{16}$$

In particular, $g(\nabla_Y X,X) = 0$ for any Y. Since $\nabla g = 0$. It implies that $dg(X,X) = 0$. Q.E.D.

B. Equivariance

Now we seek conditions for ν to be equivariant. This issue will be analyzed in the next few paragraphs. The map ν is equivariant if and only if $\nu(g\cdot x) = \mathrm{Ad}g^*(\nu(x))$. Let X be any element in $\mathfrak{g}$. The equivariance is determined by

$$\nu(g\cdot x)(X) = \nu(x)(\mathrm{Ad}\, g(X)). \tag{17}$$

The infinitesimal version of the above identity is

$$\mathcal{L}_Y \nu_X = \nu_{[V,X]}; \quad \text{equivalently} \quad \mathcal{L}_Y \nu_X - \nu_{[Y,X]} = 0. \tag{18}$$

Let $[X_b, X_a] = f_{ba}^c X_c$ be the structural equations for the algebra $\mathfrak{g}$ so that f_{ba}^c are constants. Apply the above formula to w_a and u_a, respectively, with respect to X_b, the equivariance conditions for w_a and u_a are

$$\mathcal{L}_b w_a - f_{ba}^c w_c = 0, \quad \mathcal{L}_b u_a - f_{ba}^c u_c = 0. \tag{19}$$

These are nontrivial conditions. Note that

$$\begin{aligned} d\mathcal{L}_b u_a &= \mathcal{L}_b du_a = \mathcal{L}_b \iota_a H = \iota_{\mathcal{L}_b X_a} H + \iota_a \mathcal{L}_b H \\ &= \iota_{\mathcal{L}_b X_a} H = f_{ba}^c \iota_c H = f_{ba}^c du_c = d(f_{ba}^c u_c). \end{aligned} \tag{20}$$

By Poincaré lemma, there exists a locally defined closed 1-form v_{ba} such that

$$\mathcal{L}_b u_a - f_{ba}^c u_c = v_{ba}. \tag{21}$$

Therefore, v_{ba} is the obstruction for u_a to be equivariant.

Next, note that $\mathcal{L}_a g = 0$,

$$\begin{aligned}(\mathcal{L}_b \tilde{X}_a)X &= \mathcal{L}_b(g(X_a, X)) - g(X_a, \mathcal{L}_b X)\\ &= g(\mathcal{L}_b X_a, X) + g(X_a, \mathcal{L}_b X) - g(X_a, \mathcal{L}_b X)\\ &= f_{ba}^c g(X_c, X) = f_{ba}^c \tilde{X}_c X.\end{aligned}$$

Therefore, the $\mathfrak{g}^*$-valued 1-form $w := w_a \eta^a$ is equivariant if and only if $u := u_a \eta^a$ is equivariant.

Assuming that u is equivariant. This implies that after a possible shift of u_a with respect to a closed one-form, u_a must satisfy the above equation. Note that even if u_a is equivariant, it is not unique but rather defined up to an equivariant *closed* one-form.

Next since dw_a is an (1,1)-form and if we assume that the $\partial\bar{\partial}$-lemma applies on the manifold M (see either Ref. 8, 5.11 or Ref. 9, Corollary 2.110), there is a function ν_a on M such that

$$dw_a = dd^c v_a = dId\nu_a. \tag{22}$$

Therefore, the 1-form,

$$z_a = w_a - Id\nu_a$$

is closed. In the above equation ν_a is not uniquely defined but rather it is defined up the addition of the real part of a *holomorphic* function.

As we have assumed that u_a is equivariant, w_a is equivariant. We obtain

$$Id\nu_{ba} + z_{ba} = 0, \tag{23}$$

where $\nu_{ba} = \mathcal{L}_b \nu_a - f_{ba}{}^c \nu_c$ and $z_{ba} = \mathcal{L}_b z_a - f_{ba}{}^c z_c$. Since $dz_{ba} = 0$, (23) implies that $dd^c \nu_{ba} = 0$. By $\partial\bar{\partial}$-Lemma again, ν_{ba} is a harmonic function and hence is the real part of a holomorphic function f_{ba}. If, in addition,

$$f_{ba} = \mathcal{L}_b F_a - f_{ba}{}^c F_c \tag{24}$$

for some holomorphic functions F_a, then redefining ν_a as $\nu_a - \mathrm{Re}\, F_a$ and z_a as $z_a - d\,\mathrm{Im}\, F_a$ both ν_a and z_a become equivariant. So there is a choice of u_a, such that $w_a = Id\nu_a$. Therefore, we have found an equivariant moment map $\nu: M \to \mathfrak{g}^*$.

C. Cohomology

The various conditions that we have found for the existence of a moment map in the previous section can be identified as classes in de-Rham H^*_{dR} and in H^*_δ cohomology, where δ will be defined shortly. Let δ_G be the map defining Lie algebra cohomology in the usual way.[10] In particular, for $\theta \in \mathfrak{g}^*$ and ζ, $\eta \in \mathfrak{g}$,

$$\delta_G \theta(\zeta, \eta) = -\theta([\zeta, \eta]). \tag{25}$$

Therefore, (note the convention for wedge product) in terms of structural constants with respect to the dual basis θ^a,

$$\delta_G \theta^a = -\sum_{b,c} f^a_{bc}\theta^b \otimes \theta^c$$

$$= -\sum_{b<c} f^a_{bc}(\theta^b \otimes \theta^c - \theta^c \otimes \theta^b) = -\sum_{b<c} f^a_{bc}\theta^b \wedge \theta^c = -\frac{1}{2}\sum_{b,c} f^a_{bc}\theta^b \wedge \theta^c. \tag{26}$$

In particular, $\delta_G^2 = 0$. Next for ϕ in $\Lambda^l(M)$ and X in $\mathfrak{g}$, define

$$\hat{\delta}\phi(X) := \mathcal{L}_X \phi. \tag{27}$$

Equivalently, $\hat{\delta}\phi = \mathcal{L}_a \phi \cdot \theta^a$. Then we extend this operator to

$$\delta: \Lambda^l(M) \otimes \Lambda^k \mathfrak{g}^* \otimes \mathfrak{g}^* \to \Lambda^l(M) \otimes \Lambda^{k+1}\mathfrak{g}^* \otimes \mathfrak{g}^* \tag{28}$$

as follows. If ϕ is in $\Lambda^l(M)$, θ is in $\Lambda^k\mathfrak{g}^*$ and η is in $\mathfrak{g}^*$, then define

$$\delta(\phi \cdot \theta \otimes \eta) := \hat{\delta}\phi \wedge \theta \otimes \eta + \phi \cdot \delta_G \theta \otimes \eta + (-1)^k \phi \cdot \theta \wedge \delta_G \eta. \tag{29}$$

This map generates a resolution.

$$\Lambda^l(M) \otimes \mathfrak{g}^* \xrightarrow{\delta_0} \Lambda^l(M) \otimes \Lambda^1 \mathfrak{g}^* \otimes \mathfrak{g}^* \cdots \to \Lambda^l(M) \otimes \Lambda^k \mathfrak{g}^* \otimes \mathfrak{g}^* \xrightarrow{\delta_k} \Lambda^l(M) \otimes \Lambda^{k+1}\mathfrak{g}^* \otimes \mathfrak{g}^* \cdots .$$

We claim that this resolution is a complex, i.e., $\delta_k \circ \delta_{k+1} = \delta^2 = 0$. To check, notice that

$$\begin{aligned}
\delta^2(\phi \cdot \theta \otimes \eta) &= \delta(\hat{\delta}\phi \wedge \theta) \otimes \eta + (-1)^{k+1}\hat{\delta}\phi \wedge \theta \wedge \delta_G \eta + \hat{\delta}\phi \wedge \delta_G \theta \otimes \eta \\
&\quad + \phi \cdot \delta_G^2 \phi \otimes \eta + (-1)^{k+1}\phi \cdot \delta_G \theta \wedge \delta_G \eta + (-1)^k \hat{\delta}\phi \wedge \theta \wedge \delta_G \eta \\
&\quad + (-1)^k \phi \cdot \delta_G \theta \wedge \delta_G \eta + (-1)^{2k}\phi \cdot \theta \wedge \delta_G^2 \eta \\
&= \delta(\hat{\delta}\phi \wedge \theta) \otimes \eta + \hat{\delta}\phi \wedge \delta_G \theta \otimes \eta = (\delta(\hat{\delta}\phi \wedge \theta) + \hat{\delta}\phi \wedge \delta_G \theta) \otimes \eta \\
&= (\delta(\hat{\delta}\phi) \wedge \theta - \hat{\delta}\phi \wedge \delta_G \theta + \hat{\delta}\phi \wedge \delta_G \theta) \otimes \eta \\
&= (\delta(\mathcal{L}_a \phi \theta^a)) \wedge \theta \otimes \eta \\
&= (\mathcal{L}_b \mathcal{L}_a \phi \cdot \theta^b \wedge \theta^a + \mathcal{L}_c \phi \cdot \delta_G \theta^c) \wedge \theta \otimes \eta \\
&= (\mathcal{L}_b \mathcal{L}_a \phi - \tfrac{1}{2} f^c_{ba} \mathcal{L}_c \phi) \cdot \theta^b \wedge \theta^a \wedge \theta \otimes \eta.
\end{aligned}$$

Since $[\mathcal{L}_a, \mathcal{L}_a]\phi = f^c_{ba}\mathcal{L}_c \phi$ and $f^c_{ba} = -f^c_{ab}$,

$$\mathcal{L}_b \mathcal{L}_a \phi - \tfrac{1}{2} f^c_{ba} \mathcal{L}_c \phi = \mathcal{L}_a \mathcal{L}_b \phi + \tfrac{1}{2} f^c_{ba} \mathcal{L}_c \phi = \mathcal{L}_a \mathcal{L}_b \phi - \tfrac{1}{2} f^c_{ab} \mathcal{L}_c \phi. \tag{30}$$

It shows that the term $\mathcal{L}_b \mathcal{L}_a \phi - \frac{1}{2} f^c_{ba} \mathcal{L}_c \phi$ is symmetric in the indices ab while the term $\theta^b \wedge \theta^a$ is skew symmetric in ab. It follows that $\delta(\mathcal{L}_a \phi \theta^a) = 0$ and hence $\delta^2 = 0$ as claimed.

One can now define a cohomology theory with respect to δ in the usual way and denote it with

$$H^k_\delta(\Lambda^l(M)\otimes\mathfrak{g}^*) := \frac{\ker\,\delta_k}{\text{image}\;\delta_{k-1}}. \tag{31}$$

Since δ commutes with d, one can also naturally define the cohomology groups $H^k_\delta(C^l(M)\otimes\mathfrak{g}^*)$, where $C^l(M)$ are the close l-forms on M.

A cohomology theory based on a resolution of $\mathcal{O}\otimes\mathfrak{g}^*$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on M, is similarly defined. This is possible because the group G consists of holomorphic actions. In particular, $\overline{\partial}\circ\mathcal{L}_a=\mathcal{L}_a\circ\overline{\partial}$. This cohomology is

$$H^k_\delta(\mathcal{O}\otimes\mathfrak{g}^*) := \frac{\ker\,\delta:\mathcal{O}\otimes\Lambda^k\mathfrak{g}^*\otimes\mathfrak{g}^*}{\text{image}\;\delta:\mathcal{O}\otimes\Lambda^{k-1}\mathfrak{g}^*\otimes\mathfrak{g}^*}. \tag{32}$$

Returning now in the discussion of the previous section, we have seen a necessary condition for the existence of a moment map in the KT case is that i_aH is a trivial class in $H^2_{dR}(M)$. Now we write $i_aH=du_a$ and define $u=u_a\eta^a$. This is a section of $\Lambda^1(M)\otimes\mathfrak{g}^*$. Using (26),

$$\delta u=(\delta u_a)\otimes\eta^a+\sum_c u_c\delta_G\eta^c$$

$$=\mathcal{L}_b u_a\theta^b\otimes\eta^a-\sum_{a,b,c} f^c_{ba}u_c\theta^b\otimes\eta^a \tag{33}$$

$$=\sum_{a,b}\left(\mathcal{L}_b u_a-\sum_c f^c_{ba}u_c\right)\theta^b\otimes\eta^a. \tag{34}$$

Due to (20), the 1-form part is closed. Therefore δu is an element of $C^1(M)\otimes\Lambda^1\mathfrak{g}^*\otimes\mathfrak{g}$. Obviously, it is in the kernel of δ. It defines a class in $H^1_\delta(C^1(M)\otimes\mathfrak{g}^*)$. Since u is not necessarily a closed 1-form, this class is not necessarily trivial although it is represented by δu. Due to computation of previous paragraphs, this cohomology class is the obstruction for adjusting u by a closed 1-form so that it could be equivariant.

If this class vanishes, then as we have explained $\delta(w_a\eta^a)=0$ as well. Using this and assuming that ν_a is well-defined in $w=Id\nu+z$, where $\nu=\nu_a\eta^a$ and $z=z_a\eta^a$, we have $Id\delta\nu+\delta z=0$. As we have explained in the previous section the obstruction for both z and ν to be equivariant are δz and $\delta\nu$, respectively. The last identity implies that it suffices to find the condition for $\delta\nu=0$.

Due to identity (22), $\delta w=0$ and $\mathcal{L}_aI=0$, we have $dId\delta\nu=0$. Therefore, by $\partial\overline{\partial}$-Lemma, there exists holomorphic function f_{ba} such that $\nu_{ba}=\text{Re}\,f_{ba}$. Define

$$f:=f_{ba}\theta^b\otimes\eta^a. \tag{35}$$

This is an element in $\mathcal{O}\otimes\Lambda^1\mathfrak{g}^*\otimes\mathfrak{g}^*$. The function part of δf is holomorphic as the group G consists of holomorphic actions.

However, the real part of δf is equal to $\delta\nu=0$. Therefore, δf is purely imaginary. This is possible only if $\delta f=0$. It follows that f defines a class in $H^1_\delta(\mathcal{O}\otimes\mathfrak{g}^*)$. Note that the class of f vanishes if and only if the equation $f=\delta F$ has a solution. In other words, there are solutions for Eq. (24).

Some of the conditions that we have derived above can be cast into an elegant form using equivariant cohomology.[11] In physics, it is known that the obstructions for gauging *bosonic* two-dimensional sigma models with the Wess–Zumino term[12,13] are elements of equivariant cohomology.[14] The theorem below provides sufficient conditions for KT reduction.

Theorem 3: *Let M be a strong KT manifold and G be a compact group acting on M and leaving invariant the KT structure. If the torsion three-form H admits an equivariant extension as a closed form in $EG\times_G M$, $H^1_{\bar{\delta}}(\mathcal{O}\otimes\mathfrak{g}^*)=0$ and the $\partial\bar{\partial}$-lemma applies on M, then $M//G$ is a KT manifold.*

Proof: Note that EG is the universal classifying bundle space for the group G. It can be shown that a closed three-form H in M admits an equivariant extension in $EG\times_G M$, if H is invariant under the group action of G on M and there are equivariant one-forms $\{u_a;a=1,\ldots,\dim\mathfrak{g}\}$ on M such that

$$i_a H = du_a \quad\text{and}\quad i_a u_b + i_b u_a = 0. \tag{36}$$

Of course the one-form u_a is defined up to the addition of an equivariant closed one-form v_a. Because of this, the one-form $w_a=u_a+\tilde{X}_a$ is equivariant and dw_a is an (1,1) form on M. If the $\partial\bar{\partial}$-lemma applies, then $w_a=Id\nu_a+z_a$, where ν_a is a function on M and z_a is closed one-form. It can be shown that in fact z_a is equivariant. Indeed, since w_a is equivariant and the G-action preserves the complex structure, we have

$$Id\nu_{ba}+z_{ba}=0, \tag{37}$$

where $\nu_{ba}=\mathcal{L}_b\nu_a-f_{ba}{}^c\nu_c$ and $z_{ba}=\mathcal{L}_b z_a-f_{ba}{}^c z_c$. We have seen that the obstruction for z_a and ν_a to be equivariant lies in $H^1_{\bar{\delta}}(\mathcal{O}\otimes\mathfrak{g}^*)$. Since this vanishes z_a and ν_a are equivariant. So there is a choice of u_a, such that $w_a=Id\nu_a$.

It remains to prove the transversality condition. This follows from the last condition in (36) because it implies that $i_a u_b$ is skew-symmetric and so $i_a w_b$ is the sum of a nondegenerate symmetric matrix with $i_a u_b$. Therefore ν is a G-moment map and so $M//G$ is a KT manifold. Q.E.D.

D. Moment maps on strong HKT structures

The construction of G-moment maps for the reduction of strong HKT manifolds can proceed as in the case of strong KT manifolds above. The only difference is that for each complex structure $\{I_r;r=1,2,3\}$ one gets

$$w_a=I_r d(\nu^r)_a+z^r_a, \tag{38}$$

where z^r_a are again equivariant closed one-forms provided that the obstructions in $H^1_{\bar{\delta}}(\mathcal{O}\otimes\mathfrak{g})$ vanish. In this case however it is not always possible to redefine u_a such that $w_a=I_r d(\nu^r)_a$ unless $z^1_a=z^2_a=z^3_a$. Nevertheless, we can still use the map $\nu:M\to\mathbf{R}^3\otimes\mathfrak{g}$ as defined in (38) as a moment map. This moment map is equivariant but neither transversality nor the Cauchy–Riemann conditions generically hold. Thus we have the following theorem:

Theorem 4: *Let M be a strong HKT manifold and G be a compact group acting on M and leaving invariant the HKT structure. If the torsion three-form H admits an extension as a closed form in $EG\times_G M$ such that $w_a=I_r d(\nu^r)_a$ with ν equivariant, then $M//G$ is a HKT manifold.*

Proof: The proof follows from that of reductions of strong KT manifolds and that of reductions of weak HKT manifolds. Q.E.D.

IV. POTENTIAL FUNCTIONS

Recall that if $(M,\mathcal{I},g)$ is a HKT manifold with Kähler forms ω_a, a *HKT potential* is a function ρ such that $2\omega_1=dd_1\rho+d_2d_3\rho, 2\omega_2=dd_2\rho+d_3d_1\rho, 2\omega_3=dd_3\rho+d_1d_2\rho$. In this section, we follow the methods in Ref. 15 to find a potential function on reduced space. We continue to use the notations established in Sec. II B.

Theorem 5: *Let $(M,\mathcal{I},g)$ be a HKT manifold with HKT potential function ρ. Suppose that G is a compact group of hypercomplex isometries leaving ρ invariant with moment map $\nu=(\nu_1,\nu_2,\nu_3)$ such that the tangent vectors to the orbits of G in $\nu^{-1}(0)$ are in the* $\ker(d_a\rho)$, *for* $a=1,2,3$. *Then the function ρ induces a HKT potential function on the reduced space $N=M//G$.*

Proof: Let $P:=\nu^{-1}(0)$ and $i:P\to M$ be the inclusion map. Now we first check that $i^*dd_a\rho|_{\mathcal{U}}=dd_ai^*\rho|_{\mathcal{U}}$, where $\mathcal{U}$ is defined in (5). To this end notice that

$$i^*d\rho(X^u)=d\rho(X^u),\quad i^*d\rho([X^u,Y^u])=d\rho([X^u,Y^u]),$$

and $i^*I_ad\rho(X^u)=I_ad\rho(di(X^u))=-d\rho(I_aX^u)$ because $I_ad\rho(X)=-d\rho(I_aX)$. By direct computations after restricting on points of P we have

$$\begin{aligned}(i^*dd_a\rho)(X^u,Y^u)&=di^*d_a\rho(X^u,Y^u)\\&=X^u((i^*I_ad\rho)(Y^u))-Y^u((i^*I_ad\rho)(X^u))-i^*I_ad\rho([X^u,Y^u])\\&=-X^u(d\rho(I_aY^u))+Y^u(d\rho(I_aX^u))+d\rho(I_a[X^u,Y^u])\\&=-X^u(d\rho(I_aY^u))+Y^u(d\rho(I_aX^u))+d\rho(I_a[X^u,Y^u]^u).\end{aligned}$$

The last equality is due to $d\rho(I_a[X^u,Y^u]^v)=-d_a\rho([X^u,Y^u]^v)=0$. This is true because $[X^u,Y^u]^v$ is tangent to an orbit of G and the condition in the theorem. We shall use the same argument repeatedly and implicitly in subsequent computation.

As the map ρ is G-invariant, for x in P, we may define

$$\rho_N(\pi(x)):=\rho(x), \tag{39}$$

where π is the quotient map from P onto $N=P/G$. In other words, $\pi^*\rho_N=\rho$. It follows that

$$\begin{aligned}(\pi^*dd_a\rho_N)(X^u,Y^u)&=d\pi^*d_a\rho_N(X^u,Y^u)\\&=X^u(d_a\rho_N(d\pi(Y^u)))-Y^u(d_a\rho_N(d\pi(X^u))-d_a\rho_N(d\pi([X^u,Y^u]))\\&=-X^u(d\rho_N(I_ad\pi Y^u))+Y^u(d\rho_N(I_ad\pi X^u))+d\rho_N(I_ad\pi[X^u,Y^u]))\\&=-X^u(d\rho(I_aY^u))+Y^u(d\rho(I_aX^u))+d\rho_N(d\pi(I_a[X^u,Y^u]^u))\\&=-X^u(d\rho(I_aY^u))+Y^u(d\rho(I_aX^u))+d\rho(I_a[X^u,Y^u]^u).\end{aligned}$$

It follows that $i^*dd_a\rho_{|\mathcal{U}}=\pi^*dd_a\rho_{N|\mathcal{U}}$. Similarly,

$$\begin{aligned}(\iota^*d_ad_b\rho)(X^u,Y^u)&=(I_adI_cd\rho)(di(X^u),di(Y^u))\\&=(I_adI_cd\rho)(X^u,Y^u)\\&=dI_cd\rho(I_aX^u,I_aY^u)\\&=I_aX^u(d_c\rho(I_aY^u))-I_aY^u(d_c\rho(I_aX^u))-d_c\rho([I_aX^u,I_aY^u])\\&=I_aX^u(d_c\rho(I_aY^u))-I_aY^u(d_c\rho(I_aX^u))-d_c\rho([I_aX^u,I_aY^u]^u).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\pi^*d_ad_b\rho_N)(X^u,Y^u)&=(I_adI_cd\rho_N)(d\pi(X^u),d\pi(Y^u))\\&=(dI_cd\rho_N)(I_ad\pi(X^u),I_ad\pi(Y^u))\\&=(dI_cd\rho_N)(d\pi I_a(X^u),d\pi I_a(Y^u))\\&=(\pi^*dI_cd\rho_N)(I_aX^u,I_aY^u)\\&=(d\pi^*I_cd\rho_N)(I_aX^u,I_aY^u)\\&=I_aX^u(\pi^*I_cd\rho_N(I_aY^u)-I_aY^u(\pi^*I_cd\rho_N(I_aX^u))-\pi^*I_cd\rho_N([I_aX^u,I_aY^u])\\&=I_aX^u(-d\rho_N(d\pi(I_cI_aY^u))-I_aY^u(-d\rho_N(d\pi(I_cI_aX^u))\\&\quad-d\rho_N(d\pi I_c[I_aX^u,I_aY^u])\\&=I_aX^u(I_cd\rho(I_aY^u))-I_aY^u(I_cd\rho(I_aX^u))+d\rho(d\pi I_c[I_aX^u,I_aY^u]^u)\\&=I_aX(d_c\rho(I_aY^u))-I_aY^u(d_c\rho(I_aX^u))-d_c\rho([I_aX^u,I_aY^u]^u).\end{aligned}$$

Therefore, $i^*d_bd_c\rho_{|\mathcal{U}}=\pi^*d_bd_c\rho_{N|\mathcal{U}}$ for all even permutations (abc) of (123). At the end we use the fact that the reduced Kähler forms $\overline{\omega}_a$ are characterized by the condition $(\pi^*\overline{\omega}_a)_{|\mathcal{U}}=(i^*\omega_a)_{|\mathcal{U}}$ and conclude that $2\overline{\omega}_a=dd_a\rho_N+d_bd_c\rho_N$. Q. E. D.

Remark: In the case of when the torsion vanishes the condition in the above theorem is equivalent to the one proposed by Kobak and Swann.[15] In both cases the crucial point is to ensure $i^*I_a d\rho=I_a i^*d\rho$. In both cases $d\rho(X^v)=0$ since ρ is invariant.

V. EXAMPLES

It is known that SU(3) admits invariant hypercomplex structure, constructed by Joyce. Moreover Pedersen and Poon[16] considered the deformation of this structure and succeeded to represent any "small" deformation as a hypercomplex reduced space of the space $S^1\times S^{11}$ under an appropriate S^1 action. As it is shown in Refs. 5 and 17, the space $S^1\times S^{11}$ is HKT and one can check that the S^1-actions considered in Ref. 16 Sec. 6.3 are HKT isometries. Now according to the theorem of Sec. II B we have:

Theorem 6: *Any small deformation of the invariant hypercomplex structure on $SU(3)$ admits a HKT structure.*

In the rest of this section, we will construct new HKT-metrics through a reduction process. We begin with a well-known metric, namely the Taub-NUT metric.

A. Taub-NUT metric

We use the notation of Ref. 18. Let $\mathcal{M}=\mathbb{H}\times\mathbb{H}$ with quaternionic coordinates (q,w). We identify points $(t,x,y,z)\in R^4$ with a quaternion $q\in H$: $q=t+ix+jy+kz$. The (quaternion) conjugate is $\overline{q}=t-ix-jy-kz$. The flat metric on M is

$$ds^2_{\text{flat}}=dq\,d\overline{q}+dw\,d\overline{w}. \tag{40}$$

Using left multiplication of the unit quaternions $i,j,$ and $k,$ we find the hypercomplex structure $I,J,$ and K such that

$$\begin{aligned} &Idt=dx,Idx=-dt,Idy=dz,Idz=-dy,\\ &Jdt=dy,Jdx=-dz,Jdy=-dt,Jdz=dx,\\ &Kdt=dz,Kdx=dy,Kdy=-dx,Kdz=-dt. \end{aligned}\tag{41}$$

With respect to these complex structures, the Kähler form of the flat metric $dq\,d\overline{q}$ are

$$\omega_I=dt\wedge dx+dy\wedge dz,\quad \omega_J=dt\wedge dy+dz\wedge dx,\omega_K=dt\wedge dz+dx\wedge dy. \tag{42}$$

Let G be $\mathbb{R}$, $t\in\mathbb{R}$ with the action $(q,w)\to(qe^{it},w+\lambda t)$, for λ in $\mathbb{R}$. This is a group of hyper-Kähler isometries. It generates a moment map,

$$\nu=\frac{1}{2}qi\overline{q}+\frac{\lambda}{2}(w-\overline{w}). \tag{43}$$

We write $\mathbf{r}=qi\overline{q}$, $r=|\mathbf{r}|$ and $w=y+\mathbf{y}$ so $\mathbf{r}$ and $\mathbf{y}$ are in R^3. Moreover,

$$\nu=\tfrac{1}{2}\mathbf{r}+\lambda\mathbf{y}. \tag{44}$$

Define ψ by $q=\rho e^{i\psi/2}$, where ρ is a pure quaternion. Now using the coordinates $(\psi,\mathbf{r},y,\mathbf{y})$, we write the flat metric on M as

$$ds^2_{\text{flat}}=\frac{1}{4}\left(\frac{1}{r}d\mathbf{r}^2+r(d\psi+\omega\cdot d\mathbf{r})^2\right)+dy^2+d\mathbf{y}^2, \tag{45}$$

where curl $\omega = \mathrm{grad}(1/r)$. In these coordinates the G-action is

$$(\psi,y) \to (\psi + 2t, y + \lambda t), \tag{46}$$

which leaves $\tau = \psi - 2y/\lambda$ invariant. On $\nu^{-1}(0)$, one has $\mathbf{y} = -(1/2\lambda)\,\mathbf{r}$. The induced metric in the coordinates $(\mathbf{r},\tau,y)$ on $\nu^{-1}(0)$ is

$$ds^2_{\mathrm{flat}} = \frac{1}{4}\left(\frac{1}{r}d\mathbf{r}^2 + r\left(d\tau + \frac{2}{\lambda}dy + \omega \cdot d\mathbf{r}\right)^2\right) + dy^2 + \frac{1}{4\lambda^2}d\mathbf{r}^2. \tag{47}$$

The quotient space $\nu^{-1}(0)/G$ is obtained by an orthogonal projection along the Killing vector field $\partial/\partial y$. It turns out that the quotient metric is the Taub–NUT metric,

$$ds^2_{\mathrm{TN}} = \frac{1}{4}\left(\frac{1}{r} + \frac{1}{\lambda^2}\right)d\mathbf{r}^2 + \frac{1}{4}\left(\frac{1}{r} + \frac{1}{\lambda^2}\right)^{-1}(d\tau + \omega \cdot d\mathbf{r})^2. \tag{48}$$

B. A HKT-version of the Taub–NUT metric

Given the preparation of the last section, we are now ready to consider HKT-reduction. Let h be a function of r. We consider the metric on $\mathbb{H}\backslash\{0\}\times\mathbb{H}$ given by

$$ds^2_h = \frac{h(r)}{q\bar{q}}dq\,d\bar{q} + dw\,d\bar{w} = \frac{h(r)}{r}dq\,d\bar{q} + dw\,d\bar{w}. \tag{49}$$

As $[h(r)/r]\,dq\,d\bar{q}$ is a HKT-metric on $\mathbb{H}\backslash\{0\}$ and product of HKT metrics is again a HKT metric, ds^2 is a HKT metric. Since the hypercomplex structure does not change, the group G remains hypercomplex. It is again a group of isometries. Therefore, we again use the moment maps ν generated by the action G with respect to the hyper-Kähler metric ds^2_{flat}. On $\nu^{-1}(0)$ the induced metric with respect to ds^2_h is

$$\begin{aligned}
&\frac{h}{4r}\left(\frac{1}{r}d\mathbf{r}^2 + r\left(d\tau + \frac{2}{\lambda}dy + \omega \cdot d\mathbf{r}\right)^2\right) + dy^2 + \frac{1}{4\lambda^2}d\mathbf{r}^2\\
&\quad = \frac{1}{4}\left(\frac{h}{r^2} + \frac{1}{\lambda^2}\right)d\mathbf{r}^2 + \left(1 + \frac{h}{\lambda^2}\right)dy^2\\
&\qquad + \frac{h}{2\lambda}dy \odot (d\tau + \omega \cdot d\mathbf{r}) + \frac{h}{4}(d\tau + \omega \cdot d\mathbf{r})^2.
\end{aligned} \tag{50}$$

Here we used $\alpha \odot \beta = \alpha \otimes \beta + \beta \otimes \alpha$. So $\alpha \odot \alpha = 2\alpha \otimes \alpha$.

As hyper-Kähler reduction is also obtained by orthogonal projection, the horizontal distribution $\mathcal{U}$ is defined by ker θ. Therefore, the reduced metric is obtained by taking the restriction of ds^2 on $\nu^{-1}(0)$ modulo θ or μ, where

$$\theta = \iota_{\partial/\partial y} ds^2_{\mathrm{flat}}, \qquad \mu = dy + \frac{1}{2\lambda}\,\frac{(d\tau + \omega \cdot d\mathbf{r})}{\left(\frac{1}{r} + \frac{1}{\lambda^2}\right)}. \tag{51}$$

In other words, if $\hat{g}$ is the quotient metric, then there is a 1-form α and function a on $\nu^{-1}(0)$ such that

$$ds^2_h = a\,\mu \otimes \mu + (\alpha \otimes \mu + \mu \otimes \alpha) + \hat{g}. \tag{52}$$

It follows that $\iota_{\partial/\partial y} ds^2 = a\mu + \alpha$. In our example,

$$a=1+\frac{h}{\lambda^2},\quad \alpha=\frac{1}{2\lambda}\left(h-\left(1+\frac{h}{\lambda^2}\right)\left(\frac{1}{r}+\frac{1}{\lambda^2}\right)^{-1}\right)(d\tau+\omega\cdot d\mathbf{r}). \tag{53}$$

Therefore the quotient metric is

$$\begin{aligned}&\frac{1}{4}\left(\frac{h}{r^2}+\frac{1}{\lambda^2}\right)d\mathbf{r}^2+\left(1+\frac{h}{\lambda^2}\right)dy^2+\frac{h}{2\lambda}dy\odot(d\tau+\omega\cdot d\mathbf{r})+\frac{h}{4}(d\tau+\omega\cdot d\mathbf{r})^2\\&\quad-\left(1+\frac{h}{\lambda^2}\right)\left(dy+\frac{1}{2\lambda}\frac{(d\tau+\omega\cdot d\mathbf{r})}{\frac{1}{r}+\frac{1}{\lambda^2}}\right)^2-\left(dy+\frac{1}{2\lambda}\frac{(d\tau+\omega\cdot d\mathbf{r})}{\frac{1}{r}+\frac{1}{\lambda^2}}\right)\\&\quad\odot\frac{1}{2\lambda}\left(h-\left(1+\frac{h}{\lambda^2}\right)\left(\frac{1}{r}+\frac{1}{\lambda^2}\right)^{-1}\right)(d\tau+\omega\cdot d\mathbf{r})\\&\quad=\frac{1}{4}\left(\frac{h}{r^2}+\frac{1}{\lambda^2}\right)d\mathbf{r}^2+\frac{1}{4}\left(\frac{h}{r^2}+\frac{1}{\lambda^2}\right)\left(\frac{1}{r}+\frac{1}{\lambda^2}\right)^{-2}(d\tau+\omega\cdot d\mathbf{r})^2.\end{aligned} \tag{54}$$

$$=\left(\frac{h}{r^2}+\frac{1}{\lambda^2}\right)\left(\frac{1}{r}+\frac{1}{\lambda^2}\right)^{-1}ds^2_{\mathrm{TN}}. \tag{55}$$

In particular, the quotient metric is conformally equivalent to the Taub–NUT metric, a hyper-Kähler metric.

Amongst the class of weak HKT metrics that have been constructed above, there is a strong HKT metric which is complete. This is

$$ds^2=\left(\frac{1}{r}+\frac{1}{\lambda^2}\right)ds^2_{\mathrm{TN}}. \tag{56}$$

For this metric, the function h is

$$h(r)=1+\frac{2}{\lambda^2}r+\frac{1}{\lambda^2}\left(\frac{1}{\lambda^2}-1\right)r^2. \tag{57}$$

This metric is strong HKT because the conformal factor is a harmonic function with respect to the Taub-NUT hyper-Kähler metric. The asymptotic behavior of the metric is as follows: As $r\to\infty$, the metric (56) approaches the standard metric on $S^1\times\mathbf{R}^3$. As $r\to 0$, the metric (56) approaches

$$ds^2\sim\frac{1}{r}\left(r(d\tau+\omega\cdot d\mathbf{r})^2+\frac{1}{r}d\mathbf{r}^2\right).$$

Changing back to the quaternionic coordinates q, we find that the above metric can be rewritten as

$$ds^2=\frac{1}{q\bar{q}}dq\,d\bar{q}=du^2+ds^2(S^3)$$

with $r=q\bar{q}$ and $u=\log(|q|)$. So it is the standard metric on $\mathbf{R}\times S^3$. In physics language, the metric (56) interpolates between the ten-dimensional Kaluza–Klein vacuum and the near horizon geometry of the NS5-brane.

C. A HKT-version of the Lee–Weinberg–Yi metric

We are interested in examples beyond four-real dimension. As noted in Ref. 18, a high dimension analog of the Taub-NUT metric is the Lee–Weinberg–Yi (LWY) metric. We construct a family HKT-version of this metric. Moreover, these metrics are not conformal to the LWY-metric.

We first review the construction of the LWY-metric very briefly to fix notations. We take $\mathcal{M}=\mathbb{H}^m\times\mathbb{H}^m$ with coordinates (q_a,w_a), $a=1,\ldots,m$. Let $\Lambda=(\lambda_a^b)$ be a real nondegenerate $m\times m$-matrix. Let $V=(v_a^b)$ be the inverse matrix. For $G=\mathbb{R}^m=(t_1,\ldots,t_m)$, define an action by

$$q_a\mapsto q_a e^{it_a},\quad w_a\mapsto w_a+\sum_b \lambda_a^b t_b. \tag{58}$$

With respect to the flat metric $ds^2_{\mathrm{flat}}=\Sigma_a dq_a d\overline{q_a}+\Sigma_a dw_a d\overline{w_a}$ and the hypercomplex structure defined as in (41), group G is a group hyperholomorphic isometry. The moment map

$$\nu=(\nu_1,\ldots,\nu_m):\mathcal{M}\rightarrow\mathbb{R}^m\otimes\mathbb{R}^3 \tag{59}$$

is given by

$$\nu_a=\frac{1}{2}q_a i\overline{q_a}+\frac{1}{2}\sum_b \lambda_a^b(w_b-\overline{w_b}). \tag{60}$$

Define $\mathbf{r}_a=q_a i\overline{q_a}$, $r_a=|\mathbf{r}_a|=q_a\overline{q_a}$, $\mathbf{y}_a=\frac{1}{2}(w_a-\overline{w_a})$. It follows that $w_a=y_a+\mathbf{y}_a$. Now $\mathbf{r}_a$ and $\mathbf{y}_a$ are in $\mathbb{R}^3$ and the moment map is

$$\nu_a=\frac{1}{2}\mathbf{r}_a+\sum_b \lambda_a^b\mathbf{y}_b. \tag{61}$$

Define ψ_a by $q_a=\rho_a e^{i\psi_a/2}$, where ρ_a is a pure quaternion. Now using the coordinates $(\psi_a,\mathbf{r}_a,y_a,\mathbf{y}_a)$, one may construct explicitly a hyper-Kähler metric on the quotient space in the way the Taub-NUT metric is constructed. This is the LWY-metric.

For reference in subsequent computation, we note that in these coordinates the G-action is $(\psi_a,y_a)\rightarrow(\psi_a+2t_a,y_a+\Sigma_b\lambda_a^b t_b)$. It leaves the functions

$$\tau_a=\psi_a-2\sum_b v_a^b y_b \tag{62}$$

invariant. On the level set $\nu^{-1}(0)$, $\mathbf{r}_a=-2\Sigma_b\lambda_a^b\mathbf{y}_b$. Equivalently, $\mathbf{y}_a=-\frac{1}{2}\Sigma_b v_a^b\mathbf{r}_b$.

Next, consider a new metric on $\mathcal{M}$,

$$\begin{aligned} ds^2&=\sum_a f_a(q_a\overline{q_a})dq_a d\overline{q_a}+\sum_a dw_a d\overline{w_a}\\ &=\sum_a f_a(r_a)dq_a d\overline{q_a}+\sum_a dw_a d\overline{w_a}. \end{aligned} \tag{63}$$

This is a HKT-metric. The group G is again a group of hyper-holomorphic isometries. We may use the G-moment map ν again to construct a quotient metric $\hat{g}$ with respect to ds^2.

The restriction of the metric ds^2 on $\nu^{-1}(0)$ with respect to the coordinates $(\mathbf{r}_a,\tau_a,y_a)$ is

$$\sum_a \left(\frac{f_a}{4r_a} d\mathbf{r}_a^2 + \frac{f_a r_a}{4} \left(2\sum_b v_a^b dy_b + d\tau_a + A_a \right)^2 + dy_a^2 + \frac{1}{4} \left(\sum_b v_a^b d\mathbf{r}_b \right)^2 \right)$$

$$= \frac{1}{4} \sum_{b,c} \left(\frac{\delta_b^c f_c}{r_c} + \sum_a v_a^b v_a^c \right) d\mathbf{r}_b \otimes d\mathbf{r}_c + \sum_{b,c} \left(\delta_b^c + \sum_a (f_a r_a v_a^b v_a^c) \right) dy_b \otimes dy_c$$

$$+ \frac{1}{2} \sum_{a,b} f_a r_a v_a^b dy_b \odot (d\tau_a + A_a) + \frac{1}{4} \sum_a f_a r_a (d\tau_a + A_a)^2. \tag{64}$$

To find the quotient metric $\hat{g}$, it suffices to find functions F_{ab} and 1-forms α_a such that

$$ds^2 = \sum_{a,b} F_{ab} \theta_a \otimes \theta_b + \sum_a (\theta_a \otimes \alpha_a + \alpha_a \otimes \theta_a) + \hat{g}. \tag{65}$$

Now the problem is that the Killing vector fields $\partial/\partial y_a$ generated by G on the zero level set in general are not mutually orthogonal.

From now on, we limit our discussion to the case when $\lambda_a^b = \lambda_a \delta_a^b$. Equivalently, Λ is a diagonal matrix whose nonzero entry is λ_a. Its inverse is a diagonal matrix whose non-zero entry is $v_a = 1/\lambda_a$. In this case,

$$\theta_c := \iota_{\partial/\partial y_c} ds^2 = (1 + r_c v_c^2) dy_c + \frac{1}{2} r_c v_c (d\tau_c + A_c)$$

$$= \left(1 + \frac{r_c}{\lambda_c^2} \right) dy_c + \frac{r_c}{2\lambda_c} (d\tau_c + A_c), \tag{66}$$

where $A_c := \omega(\mathbf{r}_a) \cdot d\mathbf{r}_a$. Since the vector fields $\partial/\partial y_a$ are mutually orthogonal with respect to ds^2,

$$\iota_{\partial/\partial y_c} ds^2 = (1 + r_c v_c^2) \left(\sum_a F_{ca} \theta_a + \alpha_c \right). \tag{67}$$

The restriction of the metric ds^2 on $\nu^{-1}(0)$ with respect to the coordinates $(\mathbf{r}_a, \tau_a, y_a)$ is

$$\sum_a \left(\frac{f_a}{4r_a} d\mathbf{r}_a^2 + \frac{f_a r_a}{4} (2v_a dy_a + d\tau_a + A_a)^2 + dy_a^2 + \frac{v_a^2}{4} d\mathbf{r}_a^2 \right)$$

$$= \sum_a \left(\frac{f_a}{4r_a} d\mathbf{r}_a^2 + \frac{f_a r_a}{4} \left(\frac{2}{\lambda_a} dy_a + d\tau_a + A_a \right)^2 + dy_a^2 + \frac{1}{4\lambda_a^2} d\mathbf{r}_a^2 \right)$$

$$= \sum_a \left(\tfrac{1}{4}(f_a r_a + v_a^2) d\mathbf{r}_a^2 + (1 + f_a r_a v_a^2) dy_a^2 + \tfrac{1}{2} f_a r_a v_a dy_a \odot (d\tau_a + A_a) \right.$$

$$\left. + \tfrac{1}{4} f_a r_a (d\tau_a + A_a)^2 \right).$$

Therefore,

$$\iota_{\partial/\partial y_a} ds^2 = (1 + f_a r_a v_a^2) dy_a + \tfrac{1}{2} v_a f_a r_a (d\tau_a + A_a)$$

$$= \frac{(1 + f_a r_a v_a^2)}{(1 + r_a v_a^2)} \theta_a + \tfrac{1}{2} f_a v_a r_a \left(1 - \frac{1 + f_a r_a v_a^2}{(1 + r_a v_a^2) f_a} \right) (d\tau_a + A_a) = \frac{(1 + f_a r_a v_a^2)}{(1 + r_a v_a^2)} \theta_a$$

$$+ \frac{v_a r_a}{2(1 + r_a v_a^2)} (f_a - 1)(d\tau_a + A_a).$$

It implies that the matrix (F_{ab}) is a diagonal matrix and

$$F_a = F_{aa} = \frac{(1+f_a r_a v_a^2)}{(1+r_a v_a^2)^2}, \quad \alpha_a = \frac{v_a r_a}{2(1+r_a v_a^2)^2}(f_a - 1)(d\tau_a + A_a). \tag{68}$$

Then the quotient metric is n

$$\begin{aligned} \hat{g} &= ds^2 - \sum_a (F_a \theta_a \otimes \theta_a + \theta_a \odot \alpha_a) \\ &= \frac{1}{4}\left(\frac{f_a}{r_a} + v_a^2\right) d\mathbf{r}_a^2 + \left(\frac{f_a r_a}{4} - \frac{r_a^2 v_a^2 (1+f_a r_a v_a^2)}{4(1+r_a v_a^2)^2} - \frac{v_a^2 r_a^2 (f_a - 1)}{2(1+r_a v_a^2)^2}\right)(d\tau_a + A_a)^2 \\ &= \frac{1}{4}\sum_a \left(\frac{f_a + r_a v_a^2}{1+r_a v_a^2}\right)\left(\left(\frac{1+r_a v_a^2}{r_a}\right) d\mathbf{r}_a^2 + \left(\frac{1+r_a v_a^2}{r_a}\right)^{-1}(d\tau_a + A_a)^2\right). \end{aligned} \tag{69}$$

When $f_a = 1$ for all a, we obtain a simple version of the LWY-metric,

$$ds^2_{\mathrm{LWY}} = \frac{1}{4}\sum_a \left(\left(\frac{1+r_a v_a^2}{r_a}\right) d\mathbf{r}_a^2 + \left(\frac{1+r_a v_a^2}{r_a}\right)^{-1}(d\tau_a + A_a)^2\right). \tag{70}$$

This is simple because this metric is a product metric.

In general, so long as not all the $\lambda_a = 1/v_a$ are equal, the quotient metric $\hat{g}$ is a HKT-metric. However, it is no longer conformal to the LWY-metric.

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