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Extended deformation of Kodaira surfaces

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Abstract. We present the extended Kuranishi space for a Kodaira surface as a non-trivial example to Kontsevich and Barannikov's extended deformation theory. We provide a non-trivial example of Hertling-Manin's weak Frobenius manifold. In addition, we find that Kodaira surface is its own mirror image in the sense of Merkulov. The calculations of extended deformation and the weak Frobenius structure are based on Merkulov's perturbation method. Our computation of cohomology is done in the context of compact nilmanifolds.

1. Introduction

Since the emergence of mirror symmetry in theoretical physics, there are two particularly important mathematical approaches to it. One is Kontsevich's theory of homological mirror symmetry including a theory of extended deformation [1], [7], [14]. Another is the SYZ conjecture [25].

In the homological approach, the extended deformation theory involves the full cohomology ring of complex manifolds with coefficients in polyvector fields. Its development includes a consideration of Frobenius manifolds and weak Frobenius manifolds [16]. Although the chief impetus to both the homological mirror symmetry and the SYZ conjecture is a desire to understand the mathematical rationales and validity of mirror symmetry among Calabi-Yau manifolds, the scope of Barannikov-Kontsevich theory could be significantly extended to include complex manifolds and symplectic manifolds in general. Work in such spirit could be found among Merkulov's recent work [19], [20], [21], [22] as well as in Cao and Zhou's work [4]. In particular, Merkulov develops a homotopy version of Hertling-Manin's weak Frobenius manifold theory and discovers an F_∞ functor. In [18], Merkulov considers two objects mirror image of each other if their images through the F_∞ functor agree. The key in this analysis is concerned with the deformation of the differential Gerstenhaber algebra controlling the deformations of various geometric objects. We shall briefly review the definition of differential Gerstenhaber algebra in the next section.

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In the development of extended deformation, very few examples are known [14]. Perhaps, this is not surprising because even within classical deformation theory, analysis of explicit examples of moduli space of complex manifolds has never been straightforward. In this paper, we present the extended Kuranishi space of Kodaira surfaces as a stratified superspace, and find its mirror symplectic structure in the sense of Merkulov. We shall present Kodaira surfaces as compact quotients of a nilpotent Lie group with a left-invariant complex structure. Geometrically, the Kodaira surfaces are realized as an elliptic fibration over elliptic curves. Such description allows a generalization to higher dimensions [10].

Main Theorem I. *The extended Kuranishi space of a primary Kodaira surface has four components. There are two linear components* $\mathbb{C}^{6|4}$ *and* $\mathbb{C}^{5|4}$. *It has two additional components contained in the complement of an odd linear hyperplane in* $\mathbb{C}^{5|5}$ *and in* $\mathbb{C}^{4|4}$ *respectively. It has a non-trivial weak Frobenius structure.*

The computation of Kuranishi space is completed in Theorem 16. A description of the weak Frobenius structure is found in Section 3.6. In the above theorem, we use the notion of supermanifold [15]. Our second main result is concerned with mirror symmetry of Kodaira surfaces.

Main Theorem II. *Let J be the complex structure of a primary Kodaira surface N. There exists a family of symplectic structures on the underlying smooth manifold of N such that each symplectic structure in this family is the mirror image of (N, J).*

The proof of Theorem II is based on an analysis of the differential Gerstenhaber algebras controlling the deformations of complex structures and symplectic structures. Its proof is completed in Theorem 19. The key ingredients are Lemma 13 and Lemma 18 as they allow us to translate the issue to a finite-dimensional algebraic problem.

2. Differential Gerstenhaber algebra of nilmanifolds

In this section, we recall the definition and elementary facts of differential Gerstenhaber algebras, and construct a collection of examples based on 2-step nilpotent algebra with a complex structure.

2.1. Differential Gerstenhaber algebra (DGA). Let $\mathfrak{k} = \bigoplus_{j\in\mathbb{N}} \mathfrak{k}^j$ be a complex graded vector space. If v is an element in $\mathfrak{k}^j$, it is said to be homogeneous and we denote its grading j by $\tilde{v}$.

Definition 1. A Gerstenhaber algebra is a graded vector space $\mathfrak{k}$ equipped with two product structures, a Schouten bracket $[\bullet]$ and a wedge product $\wedge$, such that the following set of axioms holds:

(L1) $[\mathfrak{k}^i \bullet \mathfrak{k}^j] \subset \mathfrak{k}^{i+j-1}$,

(L2) $[a \bullet b] = (-1)^{\tilde{a}\tilde{b}+\tilde{a}+\tilde{b}}[b \bullet a]$,

(L3) $[a \bullet [b \bullet c]] = [[a \bullet b] \bullet c] - (-1)^{\tilde{a}\tilde{b}+\tilde{a}+\tilde{b}}[b \bullet [a \bullet c]]$, (C1) $\mathfrak{f}^i \wedge \mathfrak{f}^j \subset \mathfrak{f}^{i+j}$,

(C2) $a \wedge b = (-1)^{\tilde{a}\tilde{b}} b \wedge a$,

(C3) $[a \wedge b \bullet c] = a \wedge [b \bullet c] + (-1)^{\tilde{a}\tilde{b}} b \wedge [a \bullet c]$.

If in addition, there is an operator $\bar{\partial}$ such that

(D1) $\bar{\partial}\mathfrak{f}^j \subset \mathfrak{f}^{j+1}$,

(D2) $\bar{\partial} \circ \bar{\partial} = 0$,

(D3) $\bar{\partial}[a \bullet b] = [\bar{\partial}a \bullet b] - (-1)^{\tilde{a}}[a \bullet \bar{\partial}b]$,

(D4) $\bar{\partial}(a \wedge b) = (\bar{\partial}a) \wedge b + (-1)^{\tilde{a}} a \wedge (\bar{\partial}b)$,

then the collection $(\mathfrak{f}, [\bullet], \wedge, \bar{\partial})$ is said to be a differential Gerstenhaber algebra, or simply DGA.

If we ignore the distributive law (C3), the structure of a DGA could be considered as two entities. The triple $(\mathfrak{f}, \wedge, \bar{\partial})$ with axioms (C1), (C2), (D1), (D2) and (D4) forms a differential graded algebra. The triple $(\mathfrak{f}, [\bullet], \bar{\partial})$ with axioms (L1), (L2), (L3), (D1), (D2) and (D3) forms a differential graded Lie algebra. If the grading of the algebra $\mathfrak{f}$ is reduced to $\mathbb{Z}_2$, it is the structure of odd differential Lie superalgebra [15], p. 155, [16], p. 160. (L3) should be considered as the Jacobi identity. Its alternative expression is

$$(-1)^{(\tilde{a}+1)(\tilde{c}+1)}[a \bullet [b \bullet c]] + (-1)^{(\tilde{b}+1)(\tilde{a}+1)}[b \bullet [c \bullet a]] + (-1)^{(\tilde{c}+1)(\tilde{b}+1)}[c \bullet [a \bullet b]] = 0.$$

The commutative law (L2) and the distributive law (C3) could also be expanded to include the following:

$$[a \bullet b] = (-1)^{\tilde{a}\tilde{b}+\tilde{a}+\tilde{b}}[b \bullet a] = -(-1)^{(\tilde{a}+1)(\tilde{b}+1)}[b \bullet a], \tag{1}$$

$$[a \bullet b \wedge c] = [a \bullet b] \wedge c + (-1)^{\tilde{b}\tilde{c}}[a \bullet c] \wedge b = [a \bullet b] \wedge c + (-1)^{\tilde{b}+\tilde{a}\tilde{b}} b \wedge [a \bullet c], \tag{2}$$

$$[a \wedge b \bullet c] = a \wedge [b \bullet c] + (-1)^{\tilde{a}\tilde{b}} b \wedge [a \bullet c] = a \wedge [b \bullet c] + (-1)^{\tilde{b}+\tilde{b}\tilde{c}}[a \bullet c] \wedge b. \tag{3}$$

Given a differential Gerstenhaber algebra, we may consider the cohomology with respect to the operator $\bar{\partial}$. Due to axioms (D3) and (D4), the Schouten bracket and the wedge product descend to the cohomology of the complex so that the cohomology becomes a Gerstenhaber algebra.

Definition 2. Suppose that dG and $\widehat{dG}$ are two differential Gerstenhaber algebras. A homomorphism Υ from dG to $\widehat{dG}$ is a quasi-isomorphism if the induced map from the cohomology of dG to the cohomology of $\widehat{dG}$ is an isomorphism of Gerstenhaber algebras.

2.2. Gerstenhaber algebra associated to a Lie algebra with complex structures. Suppose that $\mathfrak{g}$ is a real finite-dimensional Lie algebra with Lie bracket $[\,,]$. Let J be an integrable complex structure on $\mathfrak{g}$. In other words, $J \circ J = -\text{identity}$, and for any pair of elements A and B in $\mathfrak{g}$

$$[A,B] - [JA,JB] + J[JA,B] + J[A,JB] = 0. \tag{4}$$

The complexification of $\mathfrak{g}$ is denoted by $\mathfrak{g}_{\mathbb{C}}$. It decomposes into type $(1,0)$-vectors and $(0,1)$-vectors. i.e. $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$. The integrability assumption implies that the vector subspaces $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ are subalgebras. If ω is an element in $\wedge^k \mathfrak{g}^*$, define

$$(J\omega)(A_1,\dots,A_k) = (-1)^k \omega(JA_1,\dots,JA_k). \tag{5}$$

We obtain a complex structure on the dual vector space $\mathfrak{g}^*$. It has a corresponding decomposition $\mathfrak{g}_{\mathbb{C}}^* = \mathfrak{g}^{*(1,0)} \oplus \mathfrak{g}^{*(0,1)}$. Let $(\mathfrak{f},\wedge)$ be the exterior algebra generated by the vector space $\mathfrak{g}^{*(0,1)} \oplus \mathfrak{g}^{1,0}$. The degree-$k$ elements form a vector subspace $\mathfrak{f}^k := \bigoplus_{p+q=k} \mathfrak{g}^{*(0,q)} \otimes \mathfrak{g}^{p,0}$ where $\mathfrak{g}^{p,0}$ is the p-th exterior product of $\mathfrak{g}^{1,0}$. The space $\mathfrak{g}^{*(0,q)}$ is the q-th exterior product of its first degree counterpart.

Next, we generate the Schouten bracket $[\bullet]$ on $\mathfrak{f}$ in the following way. If U_j and V_ℓ are in $\mathfrak{g}^{1,0}$, then $[U_j \bullet V_\ell] = [U_j, V_\ell]$ is the Lie bracket of the algebra $\mathfrak{g}$. In general,

$$\begin{aligned}&[U_1 \wedge \cdots \wedge U_k \bullet V_1 \wedge \cdots \wedge V_p] \\ &\quad := \sum_{j=1}^{k} (-1)^{(k-j)} U_1 \wedge \cdots \wedge U_{j-1} \wedge U_{j+1} \wedge \cdots \wedge U_k \wedge [U_j \bullet V_1 \wedge \cdots \wedge V_p].\end{aligned}$$

For ψ in $\mathfrak{g}^{*(0,p)}$ and $V_1 \wedge \cdots \wedge V_k$ in $\mathfrak{g}^{k,0}$, define

$$[V_1 \wedge \cdots \wedge V_k \bullet \psi] := \sum_{j=1}^{k} (-1)^{k-j} V_1 \wedge \cdots \wedge V_{j-1} \wedge V_{j+1} \wedge \cdots \wedge V_k \wedge (L_{V_j}\psi).$$

If ϕ is in $\mathfrak{g}^{*(0,\ell)}$, we define $[\phi \bullet \psi] = 0$. Finally for $\phi \wedge \Xi$ in $\mathfrak{g}^{*(0,q)} \otimes \mathfrak{g}^{p,0}$ and $\psi \wedge \Theta$ in $\mathfrak{g}^{*(0,\ell)} \otimes \mathfrak{g}^{k,0}$, we define

$$\begin{aligned}&[\phi \wedge \Xi \bullet \psi \wedge \Theta] \\ &\quad := \phi \wedge [\Xi \bullet \psi] \wedge \Theta + (-1)^{(\tilde{\phi}+\tilde{\Xi})(\tilde{\psi}+\tilde{\Theta})+(\tilde{\phi}+\tilde{\Xi})+(\tilde{\psi}+\tilde{\Theta})} \psi \wedge [\Theta \bullet \phi] \wedge \Xi \\ &\qquad + (-1)^{\tilde{\psi}(\tilde{\Xi}+1)} \phi \wedge \psi \wedge [\Xi \bullet \Theta].\end{aligned} \tag{6}$$

It is straightforward to verify that the structure $(\mathfrak{f}, [\bullet], \wedge)$ forms a Gerstenhaber algebra.

2.3. Construction of a differential operator. Suppose that the complex structure J is abelian. In other words, for any A and B in $\mathfrak{g}$,

$$[JA,JB] = [A,B]. \tag{7}$$

Such complex structure is characterized by the eigenspaces $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ with respect to J being abelian subalgebras of the complexified algebra $\mathfrak{g}_{\mathbb{C}}$ [2].

Suppose in addition that the Lie algebra $\mathfrak{g}$ is a 2-step nilpotent Lie algebra. The quotient $\mathfrak{t}$ of the algebra $\mathfrak{g}$ by the center $\mathfrak{c}$ is abelian. It yields an exact sequence of algebras

$$0 \to \mathfrak{c} \to \mathfrak{g} \to \mathfrak{t} \to 0. \tag{8}$$

As vector spaces, there is a direct sum decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{c}_{\mathbb{C}}$. Since the complex structure J is abelian, the center is invariant with respect to J. Therefore, as vector spaces,

$$\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \mathfrak{c}^{0,1}, \quad \mathfrak{g}^{0,1} = \mathfrak{t}^{0,1} \oplus \mathfrak{c}^{0,1}. \tag{9}$$

Let $\{T_j, 1 \leqq j \leqq n\}$ be a basis for $\mathfrak{t}^{1,0}$.

Assume that the real dimension of the center $\mathfrak{c}$ is two-dimensional. Let W be a non-zero element in $\mathfrak{c}^{1,0}$. Then the 2-step nilpotence and (7) together imply that the structural equations of the Lie algebra are given by some constants E_{kj} and F_{kj} such that

$$[\overline{T}_j, T_k] = E_{jk} W + F_{jk} \overline{W}. \tag{10}$$

The structural constants are subjected to the constraint $\overline{F}_{kj} = -E_{jk}$.

Let $\{\omega^j, 1 \leqq j \leqq n\}$ and $\{\rho\}$ be the dual bases of $\{T_j\}$ and $\{W\}$ respectively. Then the dual structural equations are

$$d\omega^j = 0, \quad d\rho = -\sum_{j,k} E_{jk} \overline{\omega}^j \wedge \omega^k = \sum_{j,k} E_{jk} \omega^k \wedge \overline{\omega}^j. \tag{11}$$

It follows that

$$\bar{\partial}\overline{\omega}^j = 0, \quad \bar{\partial}\bar{\rho} = 0, \quad \text{and} \quad \bar{\partial}\rho = \sum_{j,k} E_{jk} \omega^k \wedge \overline{\omega}^j. \tag{12}$$

We define a linear map $\bar{\partial} : \mathfrak{g}^{1,0} \to \mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0}$ as follows [8]. For any $(1,0)$-vector A and $(0,1)$-vector $\overline{B}$, define

$$\bar{\partial}_{\overline{B}} A := [\overline{B}, A]^{1,0}. \tag{13}$$

It follows that $\bar{\partial} W = 0$ and $\bar{\partial} T_j = E_{kj} \overline{\omega}^k \wedge W$. Using the usual $\bar{\partial}$-operator on differential forms and extension by anti-derivation on polyvectors, we extend this operator to $\mathfrak{f}$. To be precise, for $\overline{\Omega} \wedge \Xi$ in $\mathfrak{g}^{*(0,q)} \otimes \mathfrak{g}^{p,0}$,

$$\bar{\partial}(\overline{\Omega} \wedge \Xi) = \bar{\partial}\overline{\Omega} \wedge \Xi + (-1)^q \overline{\Omega} \wedge \bar{\partial}\Xi. \tag{14}$$

Given the $\bar{\partial}$-operator, we have a resolution for each $\mathfrak{g}^{p,0}$:

$$0 \to \mathfrak{g}^{p,0} \to \mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{p,0} \to \cdots \to \mathfrak{g}^{*(0,k)} \otimes \mathfrak{g}^{p,0} \to \mathfrak{g}^{*(0,k+1)} \otimes \mathfrak{g}^{p,0} \to \cdots. \tag{15}$$

Lemma 3. *When the dimension of* $\mathfrak{c}^{1,0}$ *is equal to one, the above resolution is a complex.*

Proof. Since $\bar\partial\bar\Omega = 0$ for all $\bar\Omega$ in $\mathfrak{g}^{*(0,q)}$, it suffices to prove that $\bar\partial^2\Xi = 0$ for all poly-vector field Ξ. Since $\bar\partial W = 0$, we focus on $\Xi \in \mathfrak{t}^{k,0}$. By re-ordering elements in the basis, it suffices to consider $\Xi = T_1 \wedge \cdots \wedge T_k$:

$$\begin{aligned}(16)\qquad &\bar\partial\bar\partial(T_1 \wedge \cdots \wedge T_k)\\ &= \sum_{j=1}^{k} E_{kj}\bar\partial(\bar\omega^k \wedge T_1 \wedge \cdots \wedge W \wedge \cdots \wedge T_k)\\ &= -\sum_{j=1}^{k} E_{kj}\bar\omega^k \wedge \bar\partial(T_1 \wedge \cdots \wedge W \wedge \cdots \wedge T_k).\end{aligned}$$

Since the vector part of $\bar\partial T_j$ is the linear span of W due to dimension restriction and $\bar\partial W = 0$ due to the step of nilpotence, the last operation yields zero. q.e.d.

Definition 4. Let $\mathfrak{h}^{p,q}$ be the q-th cohomology of the complex (15), i.e.

$$\mathfrak{h}^{p,q} = \frac{\ker \bar\partial : \mathfrak{g}^{*(0,q)} \otimes \mathfrak{g}^{p,0} \to \mathfrak{g}^{*(0,q+1)} \otimes \mathfrak{g}^{p,0}}{\bar\partial(\mathfrak{g}^{*(0,q-1)} \otimes \mathfrak{g}^{p,0})} = \frac{\ker \bar\partial_q}{\text{image } \bar\partial_{q-1}},$$

$$\mathfrak{h}^k = \bigoplus_{p+q=k} \mathfrak{h}^{p,q}.$$

It is now straightforward to verify the following.

Proposition 5. *For any 2-step nilpotent algebra* $\mathfrak{g}$ *with an abelian complex structure* J *and real two-dimensional center*, $(\mathfrak{f}, [\bullet], \wedge, \bar\partial)$ *is a differential Gerstenhaber algebra.*

Concerned with the scope of the last theorem, we note that there are at least six series of Lie algebras qualified for the assumptions [24].

2.4. 2-Step nilmanifolds. Let $\mathfrak{g}$ be a Lie algebra as described in Proposition 5. Let G be a simply connected Lie group for the algebra $\mathfrak{g}$. Suppose that there exists a discrete subgroup Γ of G such that the quotient space $N := \Gamma\backslash G$ with respect to the left action of Γ is compact. The resulting quotient is a nilmanifold. On the group G, we use left translation of J to define a left-invariant almost complex structure on the manifold G. Due to invariance, it descends to a complex structure on the quotient space N. Since the almost complex structure is left invariant and abelian, its Nijenhuis tensor vanishes and N becomes a compact complex manifold.

The extension (8) yields a principal holomorphic fibration of N over an abelian variety with elliptic curves as fibers:

$$(17)\qquad \Psi : N = \Gamma\backslash G \to M = T^{2n}.$$

The vector field generated by W is a global holomorphic vector field trivializing the kernel of the differential of the projection map Ψ. Let $\mathscr{T}_N$ and $\mathcal{O}_N$ be the holomorphic tangent bundle and the structure sheaf respectively for the complex manifold N. The projection above yields an exact sequence of holomorphic vector bundles on N:

$$0 \to \mathfrak{c}^{1,0} \otimes \mathcal{O}_N \to \mathcal{T}_N \to \Psi^* \mathcal{T}_M \to 0. \tag{18}$$

Since $\mathfrak{c}^{1,0}$ is one-dimensional, an inspection of transition function implies that for all p, one has the exact sequence

$$0 \to \mathfrak{c}^{1,0} \otimes \Psi^* \wedge^p \mathcal{T}_M \to \wedge^{p+1} \mathcal{T}_N \to \Psi^* \wedge^{p+1} \mathcal{T}_M \to 0. \tag{19}$$

Based on the above exact sequence, a computation similar to those in the proof of Theorem 1 in [17] shows the following.

Proposition 6. *Suppose that N is a 2-step nilmanifold with abelian complex structure and complex one-dimensional center. The inclusion of the invariant differential Gerstenhaber algebra $dG(\mathfrak{g}, J)$ in the algebra $dG(N, J)$ is a quasi-isomorphism. I.e., there is a natural isomorphism $H^q(N, \wedge^p \mathcal{T}_N) \cong \mathfrak{h}^{p,q}$.*

To find representatives in cohomology classes, we take the Hermitian metric on $N = \Gamma\backslash G$ such that the frame $\{W, T_j, 1 \leqq j \leqq n\}$ is Hermitian. This choice of Hermitian metric induces a Hermitian metric on the bundle of polyvectors $\wedge^p \mathcal{T}_N$ and a Hermitian inner product on $\mathfrak{g}^{*(0,q)} \otimes \mathfrak{g}^{p,0}$. In terms of this inner product, we consider the orthogonal complement of the image $\mathfrak{g}^{*(0,q-1)} \otimes \mathfrak{g}^{p,0}$ through $\bar\partial$ in the kernel of $\bar\partial$ on $\mathfrak{g}^{*(0,q)} \otimes \mathfrak{g}^{p,0}$. Denote this space by $\bar\partial^\perp(\mathfrak{g}^{*(0,q-1)} \otimes \mathfrak{g}^{p,0})$.

Theorem 7. *For each element $\psi \wedge \Xi$ in $\mathfrak{g}^{*(0,q)} \otimes \mathfrak{g}^{p,0}$, $\bar\partial^*(\psi \wedge \Xi)$ with respect to the L_2-norm on the compact Hermitian manifold N is equal to $\bar\partial^*(\psi \wedge \Xi)$ with respect to the Hermitian inner product on the finite-dimensional vector spaces $\mathfrak{g}^{*(0,q)} \otimes \mathfrak{g}^{p,0}$. In particular, $\bar\partial^\perp(\mathfrak{g}^{*(0,q-1)} \otimes \mathfrak{g}^{p,0})$ is a space of harmonic representatives for Dolbeault cohomology $H^q(N, \wedge^p \mathcal{T}_N)$ on the compact complex manifold N.*

The idea behind the proof of this theorem is already in [10] and [17]. We do not repeat the details here.

2.5. Kodaira manifolds. Kodaira surfaces are locally trivial elliptic fibrations over elliptic curves [13]. They could be realized as the compact quotient of a nilpotent extension of a three-dimensional Heisenberg group. Below, we extend this description to all dimensions [10].

Consider the manifold R^{2n+2} with coordinates (x_j, y_j, u, v), where $1 \leqq j \leqq n$. Define a multiplication by

$$\begin{aligned}(x_j, y_j, u, v) &* (x_j', y_j', u', v') \\ &= \left(x_j + x_j', y_j + y_j', u + u' + \frac{1}{2}\sum_{j=1}^n (x_j y_j' - y_j x_j'), v + v'\right).\end{aligned} \tag{20}$$

This multiplication turns R^{2n+2} into a Lie group with the origin as identity element. This group is the product of the $(2n+1)$-dimensional Heisenberg group and the one-dimensional additive abelian group. The former is given by $v = 0$. The latter is given by $x_j = y_j = u = 0$ for all j. Let (X_j, Y_j, U, V) be the left invariant vector fields generated by left translation of $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ at the identity element. Its algebra is denoted by $\mathfrak{g}$. The non-zero structural equations are given by $[X_j, Y_j] = U$ for all $1 \leqq j \leqq n$. The algebra generated by X_j, Y_j and U is the Heisenberg algebra. The center $\mathfrak{c}$ of $\mathfrak{g}$ is spanned by U and V. The quotient of $\mathfrak{g}$ with respect to its center is denoted by $\mathfrak{t}_{2n}$.

On the algebra $\mathfrak{g}$, define an endomorphism J by

$$JX_j = Y_j, \quad JY_j = -X_j, \quad JU = V, \quad JV = -U. \tag{21}$$

It is an abelian complex structure. Let $T_j = (1/2)(X_j - iY_j)$ and $W = (1/2)(U - iV)$. Then the non-zero structural constants are $E_{jj} = -i/2$ for $1 \leqq j \leqq n$. Based on this complex structure, we construct the corresponding differential Gerstenhaber algebra $\mathfrak{f}$ as in the previous sections. The following two observations will be useful for our future computation.

Lemma 8. *The $\bar\partial$-operator of the Gerstenhaber algebra $\mathfrak{f}$ is completely generated by $\bar\partial W = 0$, $\bar\partial T_j = -\frac{i}{2}\bar\varpi^j \wedge W$, $\bar\partial\bar\varpi^j = 0$, and $\bar\partial\bar\rho = 0$.*

Lemma 9. *The Schouten bracket in the Gerstenhaber algebra $\mathfrak{f}$ is completely generated by identity $[T_j \bullet \bar\rho] = -\frac{i}{2}\bar\varpi^j$.*

Let $e_1, \ldots, e_{2n}, e_{2n+1}, e_{2n+2}$ be the standard basis for the vector space R^{2n+2}. The rank of the discrete subgroup Γ generated by them is equal to $2n+1$. Its quotient is a compact manifold. This is a Kodaira manifold. We denote it by N. Given Lemma 8 and Lemma 9, we may calculate cohomology spaces. For example, the cohomology spaces below are relevant to the deformation of generalized complex structures [11].

Lemma 10. *On a Kodaira manifold, the degree-two harmonic fields in $\mathfrak{h}$ are given as follows*:

$$\mathfrak{h}^{0,2} = \langle \mathscr{B}^j = \bar\varpi^j \wedge \bar\rho, \mathscr{B}^{ij} = \bar\varpi^i \wedge \bar\varpi^j \rangle, \quad \mathfrak{h}^{2,0} = \langle B_j = T_j \wedge W \rangle,$$

$$\mathfrak{h}^{1,1} = \left\langle \phi = \bar\rho \wedge W, \phi_k^j = \frac{1}{2}(\bar\varpi^j \wedge T_k + \bar\varpi^k \wedge T_j) \right\rangle.$$

We could also identify some elements in the center with respect to the Schouten bracket. Note that $d\bar\varpi_j = 0$ for all $1 \leqq j \leqq n$, and W commutes with all vectors, it follows that $\bar\varpi_j$ and W commute with all elements in the algebra $\mathfrak{f}$ with respect to the Schouten bracket. Furthermore, let us consider elements of the form $\bar\varpi^1 \wedge \cdots \wedge \bar\varpi^n \wedge \phi \wedge \Xi$ where $\phi \wedge \Xi \in \mathfrak{g}^{*(0,\ell)} \otimes \mathfrak{g}^{k,0}$. For any $\psi \wedge \Theta$ in $\mathfrak{g}^{*(0,q)} \otimes \mathfrak{g}^{p,0}$,

$$[\bar\varpi^1 \wedge \cdots \wedge \bar\varpi^n \wedge \phi \wedge \Xi \bullet \psi \wedge \Theta] = \bar\varpi^1 \wedge \cdots \wedge \bar\varpi^n \wedge [\phi \wedge \Xi \bullet \psi \wedge \Theta].$$

Since $[\Xi \bullet \psi]$ is contained in $\mathfrak{t}^{*(0,q)} \otimes \mathfrak{g}^{k-1,0}$, $[\Theta \bullet \phi]$ is contained in $\mathfrak{t}^{*(0,\ell)} \otimes \mathfrak{g}^{(p-1,0)}$, and $\bar\varpi^1 \wedge \cdots \wedge \bar\varpi^n$ spans the highest exterior product of $\mathfrak{t}^{*(0,1)}$, the right-hand side of the above equality is equal to zero so long as $\ell > 0$ and $q > 0$.

Lemma 11. *Elements of the form* $\bar{\omega}^1 \wedge \cdots \wedge \bar{\omega}^n \wedge \phi \wedge \Xi$ *where* $\phi \wedge \Xi \in \mathfrak{g}^{*(0,\ell)} \otimes \mathfrak{g}^{k,0}$ *with* $\ell > 0$ *are in the center of the algebra* $(\mathfrak{f}, [\bullet])$. *The subspaces* $\mathfrak{t}^{*(0,\ell)}$ *and* $\mathfrak{t}^{*(0,q)} \otimes \mathfrak{c}^{1,0}$ *are in the center of* $(\mathfrak{f}, [\bullet])$.

3. Extended deformations of Kodaira surfaces

In this section, we apply the theory of extended deformations as developed by Barannikov, Kontsevich [1] and Merkulov [18] to Kodaira surfaces.

3.1. A brief overview of extended deformation. Rigorous treatment of extended deformation theory could be found in several papers. Our principle reference is [18]. To set up the notations, we present a shortcut into the computational aspect of this theory.

On a complex manifold, there is a natural differential Gerstenhaber algebra. Its graded vector space is the space of smooth sections of bundle of $(0,q)$-forms with values in $(p,0)$-vectors.

$$\mathfrak{n} = \bigoplus_k \mathfrak{n}^k = \bigoplus_k \Big(\bigoplus_{p+q=k} C^\infty(N, \wedge^q T_N^{*(0,1)} \otimes \wedge^p T_N^{1,0}) \Big).$$

The Schouten bracket and the wedge product are fiber-wise defined as we did for the finite-dimensional vector spaces in Section 2.2 and Section 2.3.

The $\bar{\partial}$-operator is the $\bar{\partial}$-operator on differential forms twisted by the Chern connection of a Hermitian metric on holomorphic tangent bundle. The cohomology is

$$\mathfrak{h} = \bigoplus_k \mathfrak{h}^k, \quad \mathfrak{h}^k = \bigoplus_{p+q=k} H^q(N, \wedge^p \mathscr{T}_N). \tag{22}$$

In analogue with classical case, we treat elements in $\mathfrak{h}$ as infinitesimal deformations. To construct a deformation with prescribed infinitesimal deformation, we take a harmonic base $\{\theta^\alpha\}$ for $\mathfrak{h}$. Let Γ_1 be a harmonic representative with coordinates $\boldsymbol{x} = (x_\alpha)$. Here we consider $\mathfrak{h}$ as a 'supermanifold' with even and odd coordinates. A coordinate x_α is odd if and only if the grading of the corresponding harmonic element θ^α in the Gerstenhaber algebra $\mathfrak{h}$ is odd. Next we consider the formal ring $\mathfrak{n}[[\boldsymbol{x}]]$ of power series in (x_α) with coefficients in $\mathfrak{n}$. The grading of $\mathfrak{n}$ together with the parity of the coordinate functions gives elements in this ring a $\mathbb{Z}_2$-grading. For example, as an element in $\mathfrak{n}[[\boldsymbol{x}]]$, $\Gamma_1 = \sum_\alpha x_\alpha \theta^\alpha$ is always even. Now, Γ_1 is integrable if there is an even element Γ in $\mathfrak{n}[[\boldsymbol{x}]]$ with the following properties:

1. Up to degree 1, $\Gamma = \Gamma_1$. We denote it by $\Gamma \equiv_1 \Gamma_1$.

2. Γ solves the Maurer-Cartan equation.

Given any Γ_1, one could apply the Kuranishi recursive method to generate a solution to the Maurer-Cartan equation and identify the obstruction [18]. The obstruction is stored in an odd vector field $\vec{\partial}$ on the supermanifold $\mathfrak{h}$. Following Merkulov, we call the vector field $\vec{\partial}$ the Chen vector field [5]. This vector field is uniquely determined by the fact that if Γ is generated by the Kuranishi recursive method, $\vec{\partial}\Gamma$ is equal to the harmonic part of $-\bar{\partial}\Gamma - \frac{1}{2}[\Gamma \bullet \Gamma]$. Therefore, finding extended deformation is to solve the 'extended' Maurer-Cartan equation

$$\bar{\partial}\Gamma + \vec{\partial}\Gamma + \frac{1}{2}[\Gamma \bullet \Gamma] = 0 \tag{23}$$

with initial condition $\Gamma \equiv_1 \Gamma_1$.

Up to gauge equivalence the extended Kuranishi space is the zeroes of the vector field $\vec{\partial}$. Let $[\,,]$ be the usual (super) Lie bracket of vector fields on the infinite-dimensional supermanifold $\mathfrak{n}$. The gauge equivalence is due to the distribution

$$\mathscr{D} := \{[\vec{\partial}, Y] : Y \text{ is a vector field on } \mathfrak{n}\}. \tag{24}$$

In other words, the Kuranishi space is the quotient space

$$\mathscr{K} := \frac{\text{Zero } \vec{\partial}}{\mathscr{D}}. \tag{25}$$

Along the Kuranishi space, we consider the operator $\bar{\partial}_\Gamma := \bar{\partial} + [\Gamma \bullet\ ]$. It is known that the structure $(\mathfrak{n}[[\boldsymbol{x}]], [\bullet], \wedge, \bar{\partial}_\Gamma)$ is a differential Gerstenhaber algebra [18], 4.4. It means that extended deformation of complex structures gives deformation of differential Gerstenhaber algebras. A variation of the induced Gerstenhaber algebras on the cohomology level follows. To describe this variation, we take a partial derivative of the Maurer-Cartan equation with respect to the super-coordinate x_α:

$$\bar{\partial}_\Gamma\left(\frac{\partial\Gamma}{\partial x_\alpha}\right) = \bar{\partial}\left(\frac{\partial\Gamma}{\partial x_\alpha}\right) + \left[\Gamma \bullet \frac{\partial\Gamma}{\partial x_\alpha}\right] = 0. \tag{26}$$

Therefore, $\frac{\partial\Gamma}{\partial x_\alpha}$ represents a $\bar{\partial}_\Gamma$-closed class. Examining the first order terms for $\frac{\partial\Gamma}{\partial x_\alpha}$ and for all α, we realize that they span the entire cohomology.

As we have a deformed differential Gerstenhaber algebra,

$$\frac{\partial\Gamma}{\partial x_\alpha} \wedge \frac{\partial\Gamma}{\partial x_\beta}$$

is again $\bar{\partial}_\Gamma$-closed. Therefore, up to a $\bar{\partial}_\Gamma$-exact term there exist functions $\mu_\gamma^{\alpha\beta}$ such that

$$\frac{\partial\Gamma}{\partial x_\alpha} \wedge \frac{\partial\Gamma}{\partial x_\beta} = \sum_\gamma \mu_\gamma^{\alpha\beta} \frac{\partial\Gamma}{\partial x_\gamma}. \tag{27}$$

Now consider a product on the tangent bundle $\mathscr{T}_\mathscr{K}$ of the supermanifold $\mathscr{K}$ defined by

$$\frac{\partial}{\partial x_\alpha} \circ \frac{\partial}{\partial x_\beta} = \sum_\gamma \mu_\gamma^{\alpha\beta} \frac{\partial}{\partial x_\gamma}, \tag{28}$$

then the product varies over the superspace $\mathscr{K}$. This determines an associative and commutative product on the tangent sheaf $\mathscr{T}_{\mathscr{K}}$ at least along the smooth part of $\mathscr{K}$ [18], Theorem 3.6.2. It makes $(\mathscr{T}_{\mathscr{K}}, \circ)$ an F-manifold or weak Frobenius manifold [12], Corollary 4.9.2.

In the presence of obstruction, the entire computation is replaced by the operator $\bar{\partial} + \vec{\partial} + [\Gamma \bullet\ ]$ and the computation is done on the supermanifold $\mathfrak{h}$. In such case, Merkulov finds a homotopy version of F-manifolds, so called F_∞ structure on $\mathfrak{h}$.

3.2. Generalized deformations of Kodaira manifolds. As an example, we solve the Maurer-Cartan equation generated by elements in $\mathfrak{h}^2$ for Kodaira manifolds. One may consider such deformations as deformation of generalized complex structures [11].

Due to Lemma 11, the only non-zero brackets among elements in $\mathfrak{h}^2$ are

$$[\mathscr{B}^j \bullet B_k] = \bar{\partial} s_k^j, \quad \text{for all } 1 \leqq j, k \leqq n, \tag{29}$$

where

$$s_k^j = \frac{1}{2}(\overline{\omega}^j \wedge T_k - \overline{\omega}^k \wedge T_j) \quad \text{and} \quad \bar{\partial} s_k^j = \frac{i}{2}\overline{\omega}^j \wedge \overline{\omega}^k \wedge W. \tag{30}$$

It follows that the field s_k^j appears in the Kuranishi recursive formula, and hence we have to calculate its bracket with all elements in $\mathfrak{h}^2$. It turns out the only non-zero brackets are the following:

$$[s_k^j \bullet \mathscr{B}^\ell] = \frac{i}{2}\overline{\omega}^j \wedge \overline{\omega}^k \wedge \overline{\omega}^\ell, \quad [s_k^j \bullet \psi] = -\bar{\partial} s_k^j. \tag{31}$$

Since $\overline{\omega}^j \wedge \overline{\omega}^k \wedge \overline{\omega}^\ell$ commutes with all other elements in $\mathfrak{k}$ with respect to the Schouten bracket, it is now straightforward to verify the following.

Theorem 12. *Every direction Γ_1 in $\mathfrak{h}^2$ as an infinitesimal extended deformation is unobstructed. If*

$$\Gamma_1 = \sum_{j=1}^n a_j \mathscr{B}^j + \sum_{1=i<j=n} a_{ij} \mathscr{B}^{ij} + a\psi + \sum_{j,k=1}^n a_j^k \phi_k^j + \sum_{j=1}^n a^j B_i$$

and $a \neq 1$, then $(\vec{\partial}, \Gamma)$ with

$$\vec{\partial} = \vec{0}, \quad \textit{and} \quad \Gamma = \Gamma_1 - \frac{1}{1-a}\sum_{i,j=1}^n a^i a_j s_i^j \tag{32}$$

form a solution to the extended Maurer-Cartan equation with Γ_1 as initial condition.

3.3. DGA of Kodaira surfaces. In the remaining sections, we consider the extended deformation of Kodaira surfaces in all directions in $\mathfrak{h}$. We begin with finding the cohomology of all degrees.

Firstly, we consider the resolution of $\mathfrak{g}^{1,0}$. By Lemma (8), $\mathfrak{h}^{1,0} = \langle W \rangle$. Given any element $\Phi = a_1^1 \bar{\omega} \wedge T + a_1 \bar{\omega} \wedge W + a^1 \bar{\rho} \wedge T + a\bar{\rho} \wedge W$ in $\mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0}$,

$$\bar{\partial}\Phi = \bar{\partial}(a^1 \bar{\rho} \wedge T) = \frac{i}{2} a^1 \bar{\omega} \wedge \bar{\rho} \wedge W.$$

Therefore, $\ker \bar{\partial} \cap (\mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0})$ is the linear span of $\bar{\omega} \wedge T$, $\bar{\omega} \wedge W$, and $\bar{\rho} \wedge W$. Since the image of $\bar{\partial}$ in this space is spanned by $\bar{\omega} \wedge W$, $\mathfrak{h}^{1,1} = \langle \bar{\omega} \wedge T, \bar{\rho} \wedge W \rangle$. Similarly, the kernel of $\bar{\partial}$ on the space $\mathfrak{g}^{*(0,2)} \otimes \mathfrak{g}^{1,0}$ is spanned by $\bar{\omega} \wedge \bar{\rho} \wedge T$ and $\bar{\omega} \wedge \bar{\rho} \wedge W$. Since the latter is in the image of the $\bar{\partial}$-operator, $\mathfrak{h}^{1,2} = \langle \bar{\omega} \wedge \bar{\rho} \wedge T \rangle$.

For the resolution of $\mathfrak{g}^{2,0}$, since $\bar{\partial}(T \wedge W) = 0$, $\bar{\partial}\bar{\rho} = 0$ and $\bar{\partial}\bar{\omega} = 0$,

$$\mathfrak{h}^{2,0} = \langle T \wedge W \rangle, \quad \mathfrak{h}^{2,1} = \langle \bar{\omega} \wedge T \wedge W, \bar{\rho} \wedge T \wedge W \rangle, \quad \mathfrak{h}^{2,2} = \langle \bar{\omega} \wedge \bar{\rho} \wedge T \wedge W \rangle.$$

Due to Lemma 11, center of the algebra $\mathfrak{h}$ with respect to the Schouten bracket is the ideal generated by $\langle \bar{\omega} \rangle$ with respect to the exterior product. In summary, the space of harmonic representatives of the cohomology space $\mathfrak{h}$ is spanned by the following elements:

(33)

$h^{p,q}$	in the center	in the complement of the center
$h^{0,0}$	1	
$h^{0,1}$	$\bar{\omega}$	$\bar{\rho}$
$h^{0,2}$	$\bar{\omega} \wedge \bar{\rho}$	
$h^{1,0}$	W	
$h^{1,1}$	$\bar{\omega} \wedge T$	$\bar{\rho} \wedge W$
$h^{1,2}$		$\bar{\rho} \wedge T \wedge W$
$h^{2,0}$		$T \wedge W$
$h^{2,1}$	$\bar{\omega} \wedge T \wedge W$, $\bar{\omega} \wedge \bar{\rho} \wedge T$	
$h^{2,2}$	$\bar{\omega} \wedge \bar{\rho} \wedge T \wedge W$	

By taking the above classes as harmonic representatives, we are in effect choosing an inclusion of the Gerstenhaber algebra $\mathfrak{h}$ in the full differential Gerstenhaber algebra $\mathfrak{k}$. In this sense, we consider the orthogonal complement of $\mathfrak{h}^{p,q}$ in $\mathfrak{k}^{p,q}$, denoted by $(\mathfrak{h}^{p,q})^{\perp}$. This space is spanned by the following elements:

(34)

$(h^{p,q})^{\perp}$	in the center	in the complement of the center
$f^{1,0}$		T
$f^{1,1}$	$\bar{\omega} \wedge W = 2i\bar{\partial}T$	$\bar{\rho} \wedge T$
$f^{1,2}$	$\bar{\omega} \wedge \bar{\rho} \wedge W = 2i\bar{\partial}(\bar{\rho} \wedge T)$	

In the above computation, we repeatedly use the commutative law (1), distributive laws (2) and (3), and information on degree-one elements to generate all the algebraic relations among elements of higher degrees. It will be useful to summarize all the 'generating data' for the algebra.

Lemma 13. *The* DGA $dG(\mathfrak{g}, J) = (\mathfrak{f}, [\bullet], \wedge, \bar{\partial})$ *associated to the complex structure of the Kodaira surface* (N, J) *is generated as follows*:

1. *The algebra with respect to the wedge product* '$\wedge$' *is the exterior algebra generated by the degree-one elements* $\{T, W, \bar{\rho}, \bar{\omega}\}$.

2. *The sole non-zero Schouten bracket among degree-one elements is given by* $[T \bullet \bar{\rho}] = -\frac{i}{2}\bar{\omega}$.

3. *The differential on degree-one elements is determined by* $\bar{\partial}T = -\frac{i}{2}\bar{\omega} \wedge W$, $\bar{\partial}W = 0$, $\bar{\partial}\bar{\rho} = 0$, *and* $\bar{\partial}\bar{\omega} = 0$.

Proposition 14. *The odd Lie superalgebra* $(\mathfrak{h}, [\bullet]_{\mathfrak{h}})$ *is Abelian.*

Proof. Note that $\bar{\partial}W = 0$. Therefore W commutes with all elements. Moreover for any a and b in $\mathfrak{h}$, $[a \bullet b \wedge W] = [a \bullet b] \wedge W$. Therefore, the Schouten brackets among the three elements $\bar{\rho} \wedge W$, $\bar{\rho} \wedge T \wedge W$, and $T \wedge W$ are all equal to zero. We only need to consider the Schouten bracket between $\bar{\rho}$ and anyone of these three elements. Due to Lemma 8 and Lemma 9, the only non-zero brackets among elements in $\mathfrak{h}$ are

$$(35) \qquad [\bar{\rho} \bullet T \wedge W] = -\bar{\partial}T \quad \text{and} \quad [\bar{\rho} \bullet \bar{\rho} \wedge T \wedge W] = -\bar{\partial}(T \wedge \bar{\rho}).$$

As they are $\bar{\partial}$-exact, the proof is completed. q.e.d.

Finally we complete the picture by presenting all the Schouten brackets on $\mathfrak{f}$ by including T and $\bar{\rho} \wedge T$. First note that

$$(36) \qquad [T \bullet \bar{\rho} \wedge T] = -\frac{i}{2}\bar{\omega} \wedge T.$$

The rest of the brackets is presented in the next table:

(37)

$[A \bullet B]$	T	$\bar{\rho} \wedge T$
$\bar{\rho}$	$\frac{i}{2}\bar{\omega}$	$-\frac{i}{2}\bar{\omega} \wedge \bar{\rho}$
$\bar{\rho} \wedge W$	$\frac{i}{2}\bar{\omega} \wedge W = -\bar{\partial}T$	$\frac{i}{2}\bar{\omega} \wedge \bar{\rho} \wedge W = -\bar{\partial}(\bar{\rho} \wedge T)$
$T \wedge W$	0	$\frac{i}{2}\bar{\omega} \wedge T \wedge W$
$\bar{\rho} \wedge T \wedge W$	$\frac{i}{2}\bar{\omega} \wedge T \wedge W$	$-i\bar{\omega} \wedge \bar{\rho} \wedge T \wedge W$

3.4. Solving Maurer-Cartan equation. Consider an element Γ_1 in $\mathfrak{h}$. We choose coordinate functions as follows:

$$
\begin{aligned}
(38) \qquad & t_0 + t_1\overline{\omega}\wedge\overline{\rho} + t_2\overline{\rho}\wedge W + t_3\overline{\omega}\wedge T + t_4 T\wedge W + t_5\overline{\omega}\wedge\overline{\rho}\wedge T\wedge W \\
& + s_0\overline{\rho} + s_1\overline{\omega} + s_2 W + s_3\overline{\rho}\wedge T\wedge W + s_4\overline{\omega}\wedge T\wedge W + s_5\overline{\omega}\wedge\overline{\rho}\wedge T.
\end{aligned}
$$

Here $(t_0,\dots,t_5)$ are even coordinates and $(s_0,\dots,s_5)$ are odd coordinates.

In view of Table (37) and Formula (36), we conclude that the only new terms which could be added through the recursive process are T and $\overline{\rho}\wedge T$. Therefore, if Γ is generated by the Kuranishi recursive formula and Γ_1, there exist functions μ_1 and μ_2 such that

$$\Gamma = \Gamma_1 + \mu_1 T + \mu_2\overline{\rho}\wedge T.$$

Next, we combine (35), (36) and (37) to conclude that

$$
\begin{aligned}
(39) \qquad & \bar{\partial}\Gamma + \frac{1}{2}[\Gamma\bullet\Gamma] \\
& = \bar{\partial}\big((\mu_1 - \mu_1 t_2 + s_0 t_4)T\big) + \bar{\partial}\big((\mu_2 - \mu_2 t_2 + s_0 s_3)\overline{\rho}\wedge T\big) \\
& \quad - \frac{i}{2}\mu_1 s_0\overline{\omega} - \frac{i}{2}\mu_2 s_0\overline{\omega}\wedge\overline{\rho} - i\mu_2 s_3\overline{\omega}\wedge\overline{\rho}\wedge T\wedge W - \frac{i}{2}\mu_1\mu_2\overline{\omega}\wedge T \\
& \quad + \frac{i}{2}(\mu_2 t_4 - \mu_1 s_3)\overline{\omega}\wedge T\wedge W.
\end{aligned}
$$

Since $\overline{\omega}$, $\overline{\omega}\wedge\overline{\rho}$, $\overline{\omega}\wedge\overline{\rho}\wedge T\wedge W$, $\overline{\omega}\wedge T$ and $\overline{\omega}\wedge T\wedge W$ are harmonic, Γ is a solution of the Maurer-Cartan equation only if

$$\bar{\partial}\big((\mu_1 - \mu_1 t_2 + s_0 t_4)T\big) + \bar{\partial}\big((\mu_2 - \mu_2 t_2 + s_0 s_3)\overline{\rho}\wedge T\big) = 0.$$

When $|t_2| < 1$, the solution is a pair of rational functions

$$\mu_1 = -\frac{s_0 t_4}{1 - t_2} \quad \text{and} \quad \mu_2 = -\frac{s_0 s_3}{1 - t_2}.$$

In particular, $\mu_2 t_4 - \mu_1 s_3 = 0$. Therefore, when $|t_2| < 1$,

$$(40) \qquad \Gamma = \Gamma_1 - \frac{s_0 t_4}{1 - t_2}T - \frac{s_0 s_3}{1 - t_2}\overline{\rho}\wedge T.$$

To complete a solution of the extended Maurer-Cartan equation, we note that

$$
\begin{aligned}
\frac{\partial\Gamma}{\partial s_1} &= \frac{\partial\Gamma_1}{\partial s_1} = \overline{\omega}, & \frac{\partial\Gamma}{\partial t_1} &= \frac{\partial\Gamma_1}{\partial t_1} = \overline{\omega}\wedge\overline{\rho}, \\
\frac{\partial\Gamma}{\partial t_3} &= \frac{\partial\Gamma_1}{\partial t_3} = \overline{\omega}\wedge T, & \frac{\partial\Gamma}{\partial t_5} &= \frac{\partial\Gamma_1}{\partial t_5} = \overline{\omega}\wedge\overline{\rho}\wedge T\wedge W.
\end{aligned}
$$

In view of (39), we set

$$\vec{\partial} = \frac{i}{2}\mu_1 s_0 \frac{\partial}{\partial s_1} + \frac{i}{2}\mu_2 s_0 \frac{\partial}{\partial t_1} + \frac{i}{2}\mu_1\mu_2 \frac{\partial}{\partial t_3} + i\mu_2 s_3 \frac{\partial}{\partial t_5} \tag{41}$$
$$= -\frac{i}{2}\frac{s_0}{1-t_2}\left(t_4 s_0 \frac{\partial}{\partial s_1} + s_3 s_0 \frac{\partial}{\partial t_1} - \frac{t_4 s_0 s_3}{1-t_2}\frac{\partial}{\partial t_3} + 2s_3^2 \frac{\partial}{\partial t_5}\right).$$

We sum up the above observations below.

Theorem 15. *Suppose that Γ_1 is an element in the Dolbeault cohomology of polyvector fields on the Kodaira surface. Choose a harmonic basis such that Γ_1 is given by* (38). *Then Γ given by* (40) *and $\vec{\partial}$ given by* (41) *solve the extended Maurer-Cartan equation* (23).

3.5. Extended Kuranishi space. Due to (41), the Kuranishi space for a Kodaira surface has two components, namely

$$\mathscr{K}_0 = \{s_0 = 0\} \quad \text{and} \quad \mathscr{K}_1 = \{t_4 = 0, s_3 = 0\}. \tag{42}$$

When $s_0 = 0$, $\Gamma = \Gamma_1$. Since all the coefficient functions for the coordinate vector fields except $\frac{\partial}{\partial t_5}$ in $\vec{\partial}$ vanish along $s_0 = 0$ to order 2, $[\vec{\partial}, Y]$ is non-zero along $s_0 = 0$ only if $Y = \frac{\partial}{\partial s_0}$. In this case,

$$\left[\vec{\partial}, \frac{\partial}{\partial s_0}\right] = -i\frac{s_3^2}{1-t_2}\frac{\partial}{\partial t_5} \quad \operatorname{mod} s_0. \tag{43}$$

It follows that the Kuranishi space $\mathscr{K}_0$ has two strata. One is a linear space defined by $s_3 = 0$. It is isomorphic to $\mathbb{C}^{6|4}$. The other stratum is the quotient of the open set $s_3 \neq 0$ with respect to the distribution spanned by $\frac{\partial}{\partial t_5}$. Therefore, it is identified to the super-space

$$\mathscr{K}_{0,\text{generic}} = \{(t_0, t_1, t_2, t_3, t_4, s_1, s_2, s_3, s_4, s_5) \in \mathbb{C}^{5|5} : s_3 \neq 0\}. \tag{44}$$

The restriction of the distribution $\mathscr{D}$ to $\mathscr{K}_1$ is spanned by

$$2i\left[\vec{\partial}, \frac{\partial}{\partial s_3}\right] = -\frac{s_0^2}{1-t_2}\frac{\partial}{\partial t_1} \quad \text{and} \quad 2i\left[\vec{\partial}, \frac{\partial}{\partial t_4}\right] = \frac{s_0^2}{1-t_2}\frac{\partial}{\partial s_1}.$$

It has two strata. A stratum is the linear subspace of $\mathfrak{h}$ defined by $s_0 = 0$. It is isomorphic to $\mathbb{C}^{5|4}$. The component given by $s_0 \neq 0$ is the quotient of an open set in $\mathbb{C}^{5|5}$ with respect to $\frac{\partial}{\partial t_1}$ and $\frac{\partial}{\partial s_1}$. Therefore, it is an open set contained in $\mathbb{C}^{4|4}$.

Theorem 16. *The extended Kuranishi space of the Kodaira surface has four components. There are two linear components $\mathbb{C}^{6|4}$ and $\mathbb{C}^{5|4}$. It has two additional components contained in the complement of an odd linear hyperplane in $\mathbb{C}^{5|5}$ and in $\mathbb{C}^{4|4}$ respectively.*

3.6. Associative product for a weak Frobenius manifold. In this section, we calculate the associative product on the weak Frobenius manifold $\mathscr{K}_{0,\text{generic}}$ as explained in

Section 3.1. We choose this component to compute partly because of the simplicity due to the identity $\Gamma = \Gamma_1$.

Since $\dfrac{\partial \Gamma}{\partial t_0} = 1$, the tangent vector field $\dfrac{\partial}{\partial t_0}$ is the unit element. Next we calculate the wedge products among

$$\frac{\partial \Gamma}{\partial t_\alpha}, \quad \frac{\partial \Gamma}{\partial s_\beta}, \quad \alpha = 1, 2, 3, 4, \ \beta = 1, 2, 3, 4, 5.$$

For example,

$$\frac{\partial \Gamma}{\partial t_1} \wedge \frac{\partial \Gamma}{\partial t_4} = (\bar{\varpi} \wedge \bar{\rho}) \wedge (T \wedge W) = \frac{\partial \Gamma}{\partial t_5}.$$

Since we take quotient with respect to $\dfrac{\partial \Gamma}{\partial t_5}$, $\dfrac{\partial}{\partial t_1} \circ \dfrac{\partial}{\partial t_4} = 0$. Let us take two more examples. Consider

$$\frac{\partial \Gamma}{\partial s_1} \wedge \frac{\partial \Gamma}{\partial s_2} = \bar{\varpi} \wedge W.$$

Since

$$\bar{\partial}_\Gamma(T) = -\frac{i}{2}\bar{\varpi} \wedge W + \frac{i}{2} t_2 \bar{\varpi} \wedge W + \frac{i}{2} s_3 \bar{\varpi} \wedge T \wedge W,$$

$$(1 - t_2)\bar{\varpi} \wedge W \equiv s_3 \bar{\varpi} \wedge T \wedge W = s_3 \frac{\partial \Gamma}{\partial s_4} \quad \text{mod image } \bar{\partial}_\Gamma.$$

Therefore,

$$\frac{\partial \Gamma}{\partial s_1} \wedge \frac{\partial \Gamma}{\partial s_2} \equiv \frac{s_3}{1 - t_2} \frac{\partial \Gamma}{\partial s_4} \quad \text{mod image } \bar{\partial}_\Gamma,$$

and

$$\frac{\partial}{\partial s_1} \circ \frac{\partial}{\partial s_2} = \frac{s_3}{1 - t_2} \frac{\partial}{\partial s_4}.$$

Next, we consider

$$\frac{\partial \Gamma}{\partial t_1} \wedge \frac{\partial \Gamma}{\partial s_2} = \bar{\varpi} \wedge \bar{\rho} \wedge W.$$

Given $s_0 = 0$,

$$\bar{\partial}_\Gamma(\bar{\rho} \wedge T) = -\frac{i}{2}\bar{\varpi} \wedge \bar{\rho} \wedge W + \frac{i}{2} t_2 \bar{\varpi} \wedge \bar{\rho} \wedge W + \frac{i}{2} t_4 \bar{\varpi} \wedge T \wedge W$$
$$- i s_3 \bar{\varpi} \wedge \bar{\rho} \wedge T \wedge W.$$

Therefore,

$$(1-t_2)\bar{\omega}\wedge\bar{\rho}\wedge W$$
$$\equiv t_4\bar{\omega}\wedge T\wedge W - 2s_3\bar{\omega}\wedge\bar{\rho}\wedge T\wedge W = t_4\frac{\partial\Gamma}{\partial s_4} - 2s_3\frac{\partial\Gamma}{\partial t_5} \quad \text{mod image } \bar{\partial}_\Gamma.$$

Since we take quotient with respect to $\frac{\partial}{\partial t_5}$,

$$\frac{\partial\Gamma}{\partial t_1}\wedge\frac{\partial\Gamma}{\partial s_2} \equiv \frac{t_4}{1-t_2}\frac{\partial\Gamma}{\partial s_4} \quad \text{mod image } \bar{\partial}_\Gamma, \frac{\partial}{\partial t_5}.$$

Therefore,

$$\frac{\partial}{\partial t_1}\circ\frac{\partial}{\partial s_2} = \frac{t_4}{1-t_2}\frac{\partial}{\partial s_4}.$$

We apply the above computation to every pair of coordinate vectors on the Kuranishi space $\mathscr{K}_{0,\text{generic}}$. It turns out that $\frac{\partial}{\partial s_3}, \frac{\partial}{\partial s_4}, \frac{\partial}{\partial s_5}$ do not have non-trivial product. Recall that $\frac{\partial}{\partial t_0}$ is the unit element with respect to $\circ$. We put non-trivial products in a table below with respect to the ordered bases

$$\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3}, \frac{\partial}{\partial t_4}, \frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}.$$

(45)

$\circ$	$\frac{\partial}{\partial t_1}$	$\frac{\partial}{\partial t_2}$	$\frac{\partial}{\partial t_3}$	$\frac{\partial}{\partial t_4}$	$\frac{\partial}{\partial s_1}$	$\frac{\partial}{\partial s_2}$
$\frac{\partial}{\partial t_1}$	0	0	0	0	0	$\frac{t_4}{1-t_2}\frac{\partial}{\partial s_4}$
$\frac{\partial}{\partial t_2}$	0	0	0	0	$\frac{t_4}{1-t_2}\frac{\partial}{\partial s_4}$	0
$\frac{\partial}{\partial t_3}$	0	0	0	0	0	$\frac{\partial}{\partial s_4}$
$\frac{\partial}{\partial t_4}$	0	0	0	0	$\frac{\partial}{\partial s_4}$	0
$\frac{\partial}{\partial s_1}$	0	$\frac{t_4}{1-t_2}\frac{\partial}{\partial s_4}$	0	$\frac{\partial}{\partial s_4}$	0	$\frac{s_3}{1-t_2}\frac{\partial}{\partial s_4}$
$\frac{\partial}{\partial s_2}$	$\frac{t_4}{1-t_2}\frac{\partial}{\partial s_4}$	0	$\frac{\partial}{\partial s_4}$	0	$-\frac{s_3}{1-t_2}\frac{\partial}{\partial s_4}$	0

4. Mirror image of Kodaira surfaces

In this section, we continue our analysis of extended deformation of a complex structure on the Kodaira surface, but limit to harmonic 2-fields as initial conditions.

4.1. From complex to symplectic structures. Recall that $\mathfrak{h}^2$ is spanned by $\mathscr{B} = \bar{\varpi} \wedge \bar{\rho}$, $\psi = \bar{\rho} \wedge W$, $\phi = \bar{\varpi} \wedge T$, and $B = T \wedge W$. If Γ is contained in $\mathfrak{h}^2$, there are complex numbers (t_1, t_2, t_3, t_4) such that

$$\Gamma = t_1 \mathscr{B} + t_2 \psi + t_3 \phi + t_4 B. \tag{46}$$

Due to dimension limitation, the non-harmonic classical field s_j^i vanishes. Therefore, by either Theorem 12 or Theorem 15, every element in $\mathfrak{h}^2$ together with $\vec{\partial} = 0$ is a solution of the Maurer-Cartan equation.

We consider the following interpretation of the 2-fields:

$$\begin{aligned} \mathscr{B} &\in \mathfrak{g}^{*(0,2)} \subset \mathrm{End}(\mathfrak{g}^{0,1}, \mathfrak{g}^{*(0,1)}), \\ \psi, \phi &\in \mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0} \subset \mathrm{End}(\mathfrak{g}^{0,1}, \mathfrak{g}^{1,0}) \equiv \mathrm{End}(\mathfrak{g}^{*(1,0)}, \mathfrak{g}^{*(0,1)}), \\ B &\in \mathfrak{g}^{2,0} \subset \mathrm{End}(\mathfrak{g}^{*(1,0)}, \mathfrak{g}^{1,0}). \end{aligned}$$

In such context, we see that

$$\mathscr{B}(\overline{T}) = \bar{\rho}, \quad \mathscr{B}(\overline{W}) = -\bar{\varpi}, \quad B(\omega) = W, \quad B(\rho) = -T, \tag{47}$$

$$\psi(\overline{W}) = W, \quad \psi(\rho) = -\bar{\rho}, \quad \phi(\overline{T}) = T, \quad \phi(\omega) = -\bar{\varpi}. \tag{48}$$

Now consider the distribution spanned by $\overline{L} = \langle \overline{T}, \overline{W}, \omega, \rho \rangle$. It is considered as a subbundle of $(T_N \oplus T_N^*)_{\mathbb{C}}$. This is a choice of a complex structure through the distribution of $(0,1)$-vectors and their annihilators. It defines a generalized complex structure [11]. Given Γ as in (46) a new distribution $\overline{L}_\Gamma$ is defined by its graph. With respect to the ordered base $\{\overline{T}, \overline{W}, \omega, \rho, T, W, \bar{\varpi}, \bar{\rho}\}$, the distribution $\overline{L}_\Gamma$ is given by

$$\begin{pmatrix} \overline{T} + \Gamma(\overline{T}) \\ \overline{W} + \Gamma(\overline{W}) \\ \omega + \Gamma(\omega) \\ \rho + \Gamma(\rho) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & t_3 & 0 & 0 & t_1 \\ 0 & 1 & 0 & 0 & 0 & t_2 & -t_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & t_4 & -t_3 & 0 \\ 0 & 0 & 0 & 1 & -t_4 & 0 & 0 & -t_2 \end{pmatrix}. \tag{49}$$

If $t_1 = t_4 = 0$, we recover classical deformation of complex structures on Kodaira surfaces as found by Borcea [3], [10]. We may also consider the extended complex structure given by $t_2 = t_3 = 0$. There is a very special family depending on one complex parameter:

$$\overline{L}_\Gamma = \left\langle \overline{T} + \frac{1}{2} t\bar{\rho}, \overline{W} - \frac{1}{2} t\bar{\varpi}, \omega - \frac{2}{\bar{t}} W, \rho + \frac{2}{\bar{t}} T \right\rangle, \tag{50}$$

where $t \neq 0$. Let u and v be the real and imaginary part of the complex number t. Let $\omega = \alpha + i\beta$ and $\rho = \gamma + i\delta$. Define

$$\Omega = u(\alpha \wedge \gamma - \beta \wedge \delta) + v(\alpha \wedge \delta + \beta \wedge \gamma). \tag{51}$$

Then the distribution $\bar{L}_\Gamma \oplus L_\Gamma$ is the complexification of

$$\langle X + \iota_X \Omega,\ Y + \iota_Y \Omega,\ U + \iota_U \Omega,\ V + \iota_V \Omega \rangle \tag{52}$$

as a subbundle of the real direct sum $T_N \oplus T_N^*$. Therefore, the complex structure J deforms through extended deformation theory to an invariant symplectic structure Ω in the form of a generalized complex structure [11]. This family of symplectic structures is contained in the family of all invariant symplectic structures on N as we shall see next.

4.2. DGA of symplectic structures. For any set of real numbers (u_1, v_1, u_2, v_2) with $\triangle := u_1^2 + v_1^2 - u_2^2 - v_2^2 \neq 0$, we define Ω to be the closed 2-form

$$u_1(\alpha \wedge \gamma - \beta \wedge \delta) + v_1(\alpha \wedge \delta + \beta \wedge \gamma) + u_2(\alpha \wedge \gamma + \beta \wedge \delta) + v_2(\alpha \wedge \delta - \beta \wedge \gamma). \tag{53}$$

Since $\triangle \neq 0$, Ω is a symplectic form on the nilmanifold N.

Given the symplectic structure, we obtain a DGA in the standard way. Namely, a contraction with the symplectic form Ω defines an isomorphism from the tangent bundle to the cotangent bundle on the nilmanifold N. Then the Lie bracket among vector fields is carried by the inverse isomorphism to a Schouten bracket $[\bullet]_\Omega$ on de Rham algebra of differential forms on N. The package

$$dG(N, \Omega) = \Big(\bigoplus_\ell C^\infty(N, \wedge^\ell T_N^*), [\bullet]_\Omega, \wedge, d \Big)$$

is a DGA [20]. Its cohomology is the de Rham cohomology. Such construction could be limited to the space of invariant differential forms. It yields a new DGA $dG(\mathfrak{g}, \Omega) = (\wedge^* \mathfrak{g}^*, [\bullet]_\Omega, \wedge, d)$. Due to Nomizu, the inclusion of the complex of invariant differential forms of nilmanifolds in the de Rham complex induces an isomorphism of cohomology [23]. In other words, we have

Proposition 17. *The inclusion of invariant algebra $dG(\mathfrak{g}, \Omega)$ in the algebra $dG(N, \Omega)$ is a quasi-isomorphism.*

In our case, the contraction ι with Ω yields the following:

$$\iota(X) = (u_1 + u_2)\gamma + (v_1 + v_2)\delta, \qquad \iota(Y) = (v_1 - v_2)\gamma - (u_1 - u_2)\delta,$$
$$\iota(U) = -(u_1 + u_2)\alpha - (v_1 - v_2)\beta, \quad \iota(V) = -(v_1 + v_2)\alpha + (u_1 - u_2)\beta.$$

Let $\alpha' = -\frac{1}{\triangle}\iota(U)$ and $\beta' = \frac{1}{\triangle}\iota(V)$. It follows that $d\gamma = -\alpha \wedge \beta = -\triangle \alpha' \wedge \beta'$. As the only non-trivial bracket among the vectors X, Y, U, V is $[X, Y] = U$, the only non-zero bracket among the 1-forms $\alpha', \beta', \gamma, \delta$ is given by

$$[\gamma \bullet \delta]_\Omega = \alpha'. \tag{54}$$

Lemma 18. *Let Ω be the symplectic form on the Kodaira surface given by* (53). *The invariant differential Gerstenhaber algebra $dG(\mathfrak{g}, \Omega)$ is generated as follows*:

1. *The algebra with respect to the wedge product is the exterior algebra generated by the degree-one elements α', β', γ, δ.*

2. *The sole non-zero Schouten bracket among degree-one elements is given by $[\gamma \bullet \delta]_\Omega = \alpha'$.*

3. *The differential on degree-one elements is determined by $d\gamma = -\triangle \alpha' \wedge \beta'$, $d\beta' = 0$, $d\delta = 0$, and $d\alpha' = 0$.*

Theorem 19. *Let N be the Kodaira surface. There is an isomorphism*

$$\Upsilon : \left(\bigoplus_{k=0}^{2} \sum_{p+q=k} H^q(N, \wedge^p \mathscr{T}_N), [\bullet], \wedge \right) \to \left(\bigoplus_{k=0}^{2} H^k(N, \mathbb{C}), [\bullet]_\Omega, \wedge \right).$$

Proof. In view of Lemma 13 and Lemma 18, the map Υ defined by $\bar{\partial} \mapsto d$, $T \mapsto \gamma$, $W \mapsto -\triangle \beta'$, $\bar{\rho} \mapsto \delta$, $-\frac{i}{2}\varpi \mapsto \alpha'$ produces an isomorphism from $dG(\mathfrak{g}, J)$ to $dG(\mathfrak{g}, \Omega)$. The theorem then follows the quasi-isomorphisms established by Proposition 6 and Proposition 17. q.e.d.

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