

STABILITY OF ABELIAN COMPLEX STRUCTURES

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Let $M = \Gamma \backslash G$ be a nilmanifold endowed with an invariant complex structure. We prove that Kuranishi deformations of abelian complex structures are all invariant complex structures, generalizing a result in [7] for 2-step nilmanifolds. We characterize small deformations that remain abelian. As an application, we observe that at real dimension six, the deformation process of abelian complex structures is stable within the class of nilpotent complex structures. We give an example to show that this property does not hold in higher dimension.

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1. Introduction

Let $M = \Gamma \backslash G$ be a nilmanifold, i.e. a compact quotient of a simply-connected nilpotent Lie group G by a uniform discrete subgroup Γ . We assume that M has an invariant complex structure J , that is to say that J comes from a complex structure J on the Lie algebra \mathfrak{g} of G . An important class of complex structures is given by the abelian ones [1, 2], which are particular types of nilpotent complex structures considered in [5].

If G is 2-step nilpotent or equivalently M is a 2-step nilmanifold, the deformation of abelian complex structures was studied in [7], where it is proved that the Kuranishi process preserves the invariance of the deformed complex structures, at least for small deformations.

In this note, we show that this result can be generalized to k -step nilmanifolds with abelian complex structures, whatever k is.

Theorem 1.1. *Let G be a simply connected nilpotent Lie group with co-compact subgroup Γ . Then any abelian invariant complex structure on $M = \Gamma \backslash G$ has a locally complete family of deformations consisting entirely of invariant complex structures.*

For the proof, we construct a family of holomorphic fibrations which can be derived by the ascending series associated to any nilpotent Lie algebra (Sec. 2). Then, an inductive argument shows that the Dolbeault cohomology on M with coefficients in the structure and holomorphic tangent sheaf can be computed using invariant forms and invariant vectors (Lemma 3.5 and Theorem 3.6 in Sec. 3). So, like in [7], one can find harmonic representatives for the Dolbeault cohomology on M with coefficients in the holomorphic tangent sheaf. This allows to prove that application of Kuranishi’s method [6] does not take one outside the subspace of invariant tensors (Sec. 4).

It is known that deformation of abelian complex structures is not stable beginning in real dimension six [7]. We developed the condition for an infinitesimal deformation to be generated by a family of abelian complex structures in a coordinate free manner. Given the results in [7], it is not surprising to learn that an element in the first cohomology of the tangent sheaf is integrable to a family of Abelian complex structures if and only if it is infinitesimally so. Our computation also characterizes such elements in terms of the Lie algebra structure given by the Schouten bracket on the direct sum of the space of (1,0)-vectors and (0,1)-forms.

As an application of the theory developed above, we observe that at real dimension six, deformation of abelian complex structures is stable within the class of nilpotent complex structures (Sec. 5). An example shows that this phenomenon does not persist in higher dimension (Sec. 6). Due to Cleyton and Poon’s recent work [3], it is expected that the Kuranishi deformations of any invariant nilpotent complex structure consist of invariant complex structures.

2. Abelian Complex Structures

A complex structure J on a Lie algebra \mathfrak{g} is called *abelian* if $[JA, JB] = [A, B]$, for all A, B in \mathfrak{g} (or, equivalently, if the complex space of (1, 0)-vectors is an abelian algebra with respect to Lie bracket) [1, 2]. Recall also that J defines (extending it by left-translation) an invariant almost complex structure on the group G which is integrable.

The Lie groups or, more generally, the nilmanifolds with an abelian complex structure are to some extent dual to complex parallelizable nilmanifolds: indeed, in the complex parallelizable case

$$d\mathfrak{g}^{*(1,0)} \subset \mathfrak{g}^{*(2,0)},$$

and in the abelian case

$$d\mathfrak{g}^{*(1,0)} \subset \mathfrak{g}^{*(1,1)},$$

where $\mathfrak{g}^{*(p,q)}$ denotes the space of (p, q) -forms on \mathfrak{g} .

Now assume that the Lie algebra \mathfrak{g} is k -step nilpotent and set $n := \dim_{\mathbb{C}} \mathfrak{g}$. Let us use like in [5] the ascending series $\{\mathfrak{g}_\ell\}$ with

$$\begin{aligned} \mathfrak{g}_0 &= \{0\}, \\ \mathfrak{g}_1 &= \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = 0\}, \\ &\dots \\ \mathfrak{g}_\ell &= \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subset \mathfrak{g}_{\ell-1}\}, \\ &\dots \\ \mathfrak{g}_{k-1} &= \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subset \mathfrak{g}_{k-2}\}, \\ \mathfrak{g}_k &= \mathfrak{g}. \end{aligned}$$

It is apparent from definition that $\mathfrak{g}_\ell/\mathfrak{g}_{\ell-1}$ is in the center of $\mathfrak{g}/\mathfrak{g}_{\ell-1}$. Since J is abelian $J\mathfrak{g}_\ell \subseteq \mathfrak{g}_\ell$. Moreover

- (a) $\mathfrak{g}_\ell/\mathfrak{g}_{\ell-1}$ is abelian, and
- (b) $\mathfrak{g}/\mathfrak{g}_{k-1}$ is abelian.

Indeed, since $[\mathfrak{g}_\ell, \mathfrak{g}] \subset \mathfrak{g}_{\ell-1}$, we have

$$[\mathfrak{g}_\ell/\mathfrak{g}_{\ell-1}, \mathfrak{g}_\ell/\mathfrak{g}_{\ell-1}] \subset [\mathfrak{g}_\ell/\mathfrak{g}_{\ell-1}, \mathfrak{g}/\mathfrak{g}_{\ell-1}] = 0.$$

Moreover, by definition, $\mathfrak{g} = \mathfrak{g}_k$, thus $\mathfrak{g}/\mathfrak{g}_{k-1} = \mathfrak{g}_k/\mathfrak{g}_{k-1}$ is abelian.

Let $\{\omega_1, \bar{\omega}_1, \dots, \omega_n, \bar{\omega}_n\}$ be a real basis of \mathfrak{g}^* satisfying the structure equations

$$d\omega_i = \sum_{j,k \leq n} A_{ijk} \omega_j \wedge \bar{\omega}_k \quad (1 \leq i \leq n),$$

and let $\{X_1, \bar{X}_1, \dots, X_n, \bar{X}_n\}$ be the real basis of \mathfrak{g} dual to this basis of 1-forms. Without loss of generality, we can assume that the basis $\{X_i, \bar{X}_i; 1 \leq i \leq n\}$ is such that $\{X_{n-n_\ell+1}, \bar{X}_{n-n_\ell+1}, \dots, X_n, \bar{X}_n\}$ is a real basis of \mathfrak{g}_ℓ , $n_\ell = \dim_{\mathbb{C}} \mathfrak{g}_\ell$. In fact, proceeding as in the proof of [5, Theorem 12], having chosen a basis

$$\{X_1, \bar{X}_1, \dots, X_{n-n_{k-1}}, \bar{X}_{n-n_{k-1}}\}$$

of the Lie algebra $\mathfrak{g}/\mathfrak{g}_{k-1}$, we complete it to a basis

$$\{X_1, \bar{X}_1, \dots, X_{n-n_{k-1}}, \bar{X}_{n-n_{k-1}}, X_{n-n_{k-1}+1}, \bar{X}_{n-n_{k-1}+1}, \dots, X_{n-n_{k-2}}, \bar{X}_{n-n_{k-2}}\}$$

of the Lie algebra $\mathfrak{g}/\mathfrak{g}_{k-2}$ and so on, until we have a basis

$$\{X_1, \bar{X}_1, \dots, X_n, \bar{X}_n\}$$

of the Lie algebra \mathfrak{g} . Thus,

$$\{X_{n-n_\ell+1}, \bar{X}_{n-n_\ell+1}, \dots, X_n, \bar{X}_n\}$$

is a basis for \mathfrak{g}_ℓ , and $\{\omega_i, \bar{\omega}_i; 1 \leq i \leq n - n_\ell\}$ determines the quotient Lie algebra $\mathfrak{g}/\mathfrak{g}_\ell$. Moreover,

Lemma 2.1. *The forms $\bar{\omega}_1, \dots, \bar{\omega}_{n-n_\ell}, \dots, \bar{\omega}_n$ are all $\bar{\partial}$ -closed.*

3. Cohomology Theory

As in [4, 5, 7], we need to identify Dolbeault and Lie algebra cohomology, in the spirit of [8]. To this aim we construct a chain of fibrations with tori as fibres which are associated to the ascending series considered above.

3.1. Fibrations

Let G be a simply connected nilpotent Lie group with Lie algebra \mathfrak{g} , $M = \Gamma \backslash G$ be the corresponding nilmanifold, where Γ is a discrete uniform subgroup of G . Let J be an abelian complex structure on \mathfrak{g} .

Let $\{\mathfrak{g}_\ell\}$ be the ascending series of Sec. 2 associated with \mathfrak{g} . Then we have the following exact sequence of Lie algebras.

$$\begin{array}{ccccccc}
 0 & \rightarrow & \mathfrak{g}_{k-1}/\mathfrak{g}_{k-2} & \rightarrow & \mathfrak{g}/\mathfrak{g}_{k-2} & \rightarrow & \mathfrak{g}/\mathfrak{g}_{k-1} \rightarrow 0 \\
 & & \text{abelian} & & & & \text{abelian} \\
 & & \dots & & & & \\
 0 & \rightarrow & \mathfrak{g}_\ell/\mathfrak{g}_{\ell-1} & \rightarrow & \mathfrak{g}/\mathfrak{g}_{\ell-1} & \rightarrow & \mathfrak{g}/\mathfrak{g}_\ell \rightarrow 0 \\
 & & \text{abelian} & & & & \\
 & & \dots & & & & \\
 0 & \rightarrow & \mathfrak{g}_1 & \rightarrow & \mathfrak{g} & \rightarrow & \mathfrak{g}/\mathfrak{g}_1 \rightarrow 0. \\
 & & \text{abelian} & & & &
 \end{array}$$

Remark 3.1. We have the following splittings as vector spaces

$$\begin{aligned}
 \mathfrak{g}/\mathfrak{g}_\ell &= \mathfrak{g}/\mathfrak{g}_{k-1} \oplus \dots \oplus \mathfrak{g}_{\ell+2}/\mathfrak{g}_{\ell+1} \oplus \mathfrak{g}_{\ell+1}/\mathfrak{g}_\ell, \\
 \mathfrak{g}/\mathfrak{g}_{\ell+1} &= \mathfrak{g}/\mathfrak{g}_{k-1} \oplus \dots \oplus \mathfrak{g}_{\ell+2}/\mathfrak{g}_{\ell+1}.
 \end{aligned}$$

In particular, $\mathfrak{g}/\mathfrak{g}_\ell = \mathfrak{g}/\mathfrak{g}_{\ell+1} \oplus \mathfrak{g}_{\ell+1}/\mathfrak{g}_\ell$ as a vector space.

Let G^ℓ ($\ell = 0, \dots, k-1$) and $G^{\ell, \ell-1}$ denote the simply connected nilpotent Lie group corresponding to $\mathfrak{g}/\mathfrak{g}_\ell$ and $\mathfrak{g}_\ell/\mathfrak{g}_{\ell-1}$ respectively (note that $G^0 = G$). Then we have the surjective homomorphism (which is actually holomorphic)

$$p_\ell : G^{\ell-1} \rightarrow G^\ell$$

with fibre the abelian Lie group $G^{\ell, \ell-1}$.

Let Γ be the uniform discrete subgroup Γ of G . Since \mathfrak{g}_1 is a rational subalgebra of \mathfrak{g} , one can see that $p_1(\Gamma)$ is uniform in G^1 [10]. So we have the holomorphic fibration

$$\pi_1 : \Gamma \backslash G \rightarrow p_1(\Gamma) \backslash G^1,$$

whose fibre is a torus. Inductively we have holomorphic fibrations with a torus as fibre

$$\pi_\ell : \Gamma^{\ell-1} \backslash G^{\ell-1} \rightarrow p_\ell(\Gamma^{\ell-1}) \backslash G^\ell,$$

if we set $\Gamma^\ell := p_\ell(\Gamma^{\ell-1})$.

For the sake of simplicity we denote $M_\ell := \Gamma^\ell \backslash G^\ell$. So we have a sequence of nilmanifolds M_ℓ ($\ell = 0, \dots, k - 1$) and holomorphic fibrations

$$\pi_\ell : M_{\ell-1} \rightarrow M_\ell,$$

whose fibres are tori T_ℓ , with abelian Lie algebras $\mathfrak{t}_\ell := \mathfrak{g}_\ell / \mathfrak{g}_{\ell-1}$. Note that M_{k-1} is a torus as well, so that the “last” fibration $\pi_{k-1} : M_{k-2} \rightarrow M_{k-1}$ has a torus both as fibre and as base.

3.2. Lie algebra cohomology

Let L be a Lie group and \mathfrak{l} be its Lie algebra endowed with a complex structure J . Then the complexified Lie algebra has a type decomposition $\mathfrak{l}_\mathbb{C} = \mathfrak{l}^{1,0} \oplus \mathfrak{l}^{0,1}$. These are all spaces of left-invariant vectors on L . The definitions are extended to invariant (p, q) -forms in the standard way. For instance, $\wedge^k \mathfrak{l}_\mathbb{C}^{*(0,1)} = \mathfrak{l}^{*(0,k)}$ is the space of G -invariant $(0, k)$ -forms.

As in [7], we use a linear operator $\bar{\partial}$ on $(0, 1)$ -vectors as follows. For any $(1, 0)$ -vector V and $(0, 1)$ -vector \bar{U} , set

$$\bar{\partial}_{\bar{U}} V := [\bar{U}, V]^{1,0}.$$

We obtain a linear map

$$\bar{\partial} : \mathfrak{l}^{1,0} \rightarrow \mathfrak{l}^{*(0,1)} \otimes \mathfrak{l}^{1,0}.$$

In view of Lemma 2.1, when the complex structure J is abelian, we extend this definition to a linear map on $\mathfrak{l}^{*(0,k)} \otimes \mathfrak{l}^{1,0}$ by setting

$$\bar{\partial}_k(\bar{\omega} \otimes V) = (-1)^k \bar{\omega} \wedge \bar{\partial} V,$$

where $\omega \in \mathfrak{l}^{*(0,k)}$ and $V \in \mathfrak{l}^{1,0}$. We have a sequence

$$0 \rightarrow \mathfrak{l}^{1,0} \rightarrow \mathfrak{l}^{*(0,1)} \otimes \mathfrak{l}^{1,0} \rightarrow \dots \rightarrow \mathfrak{l}^{*(0,k-1)} \otimes \mathfrak{l}^{1,0} \xrightarrow{\bar{\partial}_{k-1}} \mathfrak{l}^{*(0,k)} \otimes \mathfrak{l}^{1,0} \xrightarrow{\bar{\partial}_k} \dots$$

Then we have (see [7, Lemma 3])

Lemma 3.2. *The above sequence is a complex, i.e. $\bar{\partial}_k \circ \bar{\partial}_{k-1} = 0$ for all $k \geq 1$.*

Accordingly we recall the following

Definition 3.3. Define $H_{\bar{\partial}}^k(\mathfrak{l}^{1,0})$ to be the k th cohomology of the above complex; more precisely,

$$H_{\bar{\partial}}^k(\mathfrak{l}^{1,0}) = \frac{\ker \bar{\partial}_k}{\text{Im } \bar{\partial}_{k-1}} = \frac{\ker(\bar{\partial}_k : \mathfrak{l}^{*(0,k)} \otimes \mathfrak{l}^{1,0} \rightarrow \mathfrak{l}^{*(0,k+1)} \otimes \mathfrak{l}^{1,0})}{\bar{\partial}_{k-1}(\mathfrak{l}^{*(0,k-1)} \otimes \mathfrak{l}^{1,0})}.$$

In the sequel, we will use this cohomology both to the k -step nilpotent Lie algebra \mathfrak{g} and the quotients $\mathfrak{g}/\mathfrak{g}_\ell$. We shall also need a “relative” version of Definition 3.3 (see the proof of Theorem 3.6).

3.3. Dolbeault cohomology

We consider again the fibrations

$$\pi_\ell : M_{\ell-1} \rightarrow M_\ell, \quad \ell = 0, \dots, k-1.$$

The fibre of π_ℓ is a torus T_ℓ , whose abelian Lie algebra is $\mathfrak{t}_\ell := \mathfrak{g}_\ell / \mathfrak{g}_{\ell-1}$.

Recall that the “last” fibration $\pi_{k-1} : M_{k-2} \rightarrow M_{k-1}$ has a torus both as fibre and as base.

Our goal is to generalize [7, Lemmas 4 and 5 and Theorem 1]. For [7, Lemma 5 and Theorem 1], the idea is to start with the “last” fibration which is analogue to the fibration considered in [7], and go on inductively. At any step the basis of the fibre bundle is not a torus but it has “good properties” (since it is the total space of the fibre bundle in the previous step).

For generalizing [7, Lemma 4], there is no difficulty, since all we need is the basis constructed in Sec. 2.

Lemma 3.4. *Let $\mathcal{O}_{M_{\ell-1}}$ and $\Theta_{M_{\ell-1}}$ be the structure sheaf and the tangent sheaf of $M_{\ell-1}$. For $j \geq 1$, the direct image sheaves with respect to π_ℓ are*

$$\begin{aligned} R^j \pi_{\ell*} \mathcal{O}_{M_{\ell-1}} &= \wedge^j \mathfrak{t}_\ell^{*(0,1)} \otimes \mathcal{O}_{M_\ell} = \mathfrak{t}_\ell^{*(0,j)} \otimes \mathcal{O}_{M_\ell}, \\ R^j \pi_{\ell*} \pi_\ell^* \Theta_{M_\ell} &= \wedge^j \mathfrak{t}_\ell^{*(0,1)} \otimes \Theta_{M_\ell} = \mathfrak{t}_\ell^{*(0,j)} \otimes \Theta_{M_\ell}. \end{aligned}$$

As a start, let us consider the structure sheaf of a general M_ℓ :

Lemma 3.5. *Let $\ell = 0, \dots, k-1$ and \mathcal{O}_{M_ℓ} be the structure sheaf M_ℓ . Then*

$$H^j(M_\ell, \mathcal{O}_{M_\ell}) = \wedge^j (\mathfrak{g} / \mathfrak{g}_\ell)^{*(0,1)} = (\mathfrak{g} / \mathfrak{g}_\ell)^{*(0,j)}.$$

Proof. We prove this lemma by induction, beginning with $\ell = k-1$ and finishing at $\ell = 0$. Note that M_{k-1} is a torus.

1st inductive step. Consider the “last” fibration $\pi_{k-1} : M_{k-2} \rightarrow M_{k-1}$ and the corresponding exact sequence of Lie algebras

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathfrak{t}_{k-1} = \mathfrak{g}_{k-1} / \mathfrak{g}_{k-2} & \rightarrow & \mathfrak{g} / \mathfrak{g}_{k-2} & \rightarrow & \mathfrak{g} / \mathfrak{g}_{k-1} \rightarrow 0. \\ & & \text{abelian} & & & & \text{abelian} \end{array}$$

Using the Leray spectral sequence with respect to the $\bar{\partial}$ -operator and the holomorphic projection π_{k-1} , we have

$$E_2^{p,q} = H^p(M_{k-1}, R^q \pi_{k-1*} \mathcal{O}_{M_{k-2}}), \quad E_\infty^{p,q} \Rightarrow H^{p+q}(M_{k-2}, \mathcal{O}_{M_{k-2}}).$$

From the previous lemma, when $q \geq 1$,

$$\begin{aligned} E_2^{p,q} &= H^p(M_{k-1}, \wedge^q \mathfrak{t}_{k-1}^{*(0,1)} \otimes \mathcal{O}_{M_{k-1}}) = \wedge^q \mathfrak{t}_{k-1}^{*(0,1)} \otimes H^p(M_{k-1}, \mathcal{O}_{M_{k-1}}) \\ &= \wedge^q \mathfrak{t}_{k-1}^{*(0,1)} \otimes \wedge^p (\mathfrak{g} / \mathfrak{g}_{k-1})^{*(0,1)}. \end{aligned}$$

Every element in $E_2^{p,q}$ is a linear combination of the tensor products of vertical $(0, q)$ -forms and $(0, p)$ -forms lifted from the base. Since these forms are globally

defined and the differential d_2 is generated by the $\bar{\partial}$ -operator, we have $d_2 = 0$. It follows that the Leray spectral sequence degenerates at the E_2 -level. Therefore, since by Remark 3.1, $\mathfrak{g}/\mathfrak{g}_{k-2} = \mathfrak{g}/\mathfrak{g}_{k-1} \oplus \mathfrak{g}_{k-1}/\mathfrak{g}_{k-2}$ as a vector space,

$$\begin{aligned} H^j(M_{k-2}, \mathcal{O}_{M_{k-2}}) &= \bigoplus_{p+q=j} E_2^{p,q} = \bigoplus_{p+q=j} \Lambda^q \mathfrak{t}_{k-1}^{*(0,1)} \otimes \Lambda^p(\mathfrak{g}/\mathfrak{g}_{k-1})^{*(0,1)} \\ &= \bigoplus_{p+q=j} \Lambda^q(\mathfrak{g}_{k-1}/\mathfrak{g}_{k-2})^{*(0,1)} \otimes \Lambda^p(\mathfrak{g}/\mathfrak{g}_{k-1})^{*(0,1)} \\ &= \Lambda^j(\mathfrak{g}/\mathfrak{g}_{k-2})^{*(0,1)} = (\mathfrak{g}/\mathfrak{g}_{k-2})^{*(0,j)}. \end{aligned}$$

General inductive step. Consider the fibration $\pi_{\ell+1} : M_\ell \rightarrow M_{\ell+1}$ and the corresponding exact sequence of Lie algebras

$$0 \rightarrow \mathfrak{t}_{\ell+1} = \mathfrak{g}_{\ell+1}/\mathfrak{g}_\ell \rightarrow \mathfrak{g}/\mathfrak{g}_\ell \rightarrow \mathfrak{g}/\mathfrak{g}_{\ell+1} \rightarrow 0.$$

abelian

Using again the Leray spectral sequence and the same arguments

$$E_2^{p,q} = \Lambda^q \mathfrak{t}_{\ell+1}^{*(0,1)} \otimes H^p(M_{\ell+1}, \mathcal{O}_{M_{\ell+1}}).$$

By inductive hypothesis $H^p(M_{\ell+1}, \mathcal{O}_{M_{\ell+1}}) = \Lambda^p(\mathfrak{g}/\mathfrak{g}_{\ell+1})^{*(0,1)}$, so

$$\begin{aligned} H^j(M_\ell, \mathcal{O}_{M_\ell}) &= \bigoplus_{p+q=j} E_2^{p,q} = \bigoplus_{p+q=j} \Lambda^q \mathfrak{t}_{\ell+1}^{*(0,1)} \otimes \Lambda^p(\mathfrak{g}/\mathfrak{g}_{\ell+1})^{*(0,1)} \\ &= \bigoplus_{p+q=j} \Lambda^q(\mathfrak{g}_{\ell+1}/\mathfrak{g}_\ell)^{*(0,1)} \otimes \Lambda^p(\mathfrak{g}/\mathfrak{g}_{\ell+1})^{*(0,1)} \\ &= \Lambda^j(\mathfrak{g}/\mathfrak{g}_\ell)^{*(0,1)} = (\mathfrak{g}/\mathfrak{g}_\ell)^{*(0,j)}. \end{aligned} \quad \square$$

Theorem 3.6. *Let M be a k -step nilmanifold with an abelian complex structure. Let $\ell = 0, \dots, k - 1$ and Θ_{M_ℓ} be the tangent sheaf M_ℓ . Then*

$$H^j(M_\ell, \Theta_{M_\ell}) \cong H_{\bar{\partial}}^j((\mathfrak{g}/\mathfrak{g}_\ell)^{(1,0)}).$$

In particular, there is a natural isomorphism $H^j(M, \Theta_M) \cong H_{\bar{\partial}}^j(\mathfrak{g}^{1,0})$.

Proof. Again, we prove this lemma by induction, beginning with $\ell = k - 1$ and finishing at $\ell = 0$.

1st inductive step. Consider the “last” fibration $\pi_{k-1} : M_{k-2} \rightarrow M_{k-1}$ and use the Leray spectral sequence of $\pi_{k-1}^* \Theta_{M_{k-1}}$ because both the base and the fiber are tori. We are in same setting as in [7, Lemma 5 and Theorem 1]. Thus, we have an analogue of [7, Lemma 5], i.e.

$$H^j(M_{k-2}, \pi_{k-1}^* \Theta_{M_{k-1}}) = \Lambda^j(\mathfrak{g}/\mathfrak{g}_{k-2})^{*(0,1)} \otimes (\mathfrak{g}/\mathfrak{g}_{k-1})^{1,0}.$$

Using the exact sequence

$$0 \rightarrow (\mathfrak{g}_{k-1}/\mathfrak{g}_{k-2})^{1,0} \otimes \mathcal{O}_{M_{k-2}} \rightarrow \Theta_{M_{k-2}} \rightarrow \pi_{k-1}^* \Theta_{M_{k-1}} \rightarrow 0$$

as in [7, Theorem 1], we get

$$H^j(M_{k-2}, \Theta_{M_{k-2}}) \cong H^j_{\bar{\partial}}((\mathfrak{g}/\mathfrak{g}_{k-2})^{(1,0)}).$$

2nd inductive step. Consider the fibration $\pi_{k-2} : M_{k-3} \rightarrow M_{k-2}$. Now the basis is not necessarily a torus. The Leray spectral sequence of $\pi_{k-2}^* \Theta_{M_{k-2}}$ yields

$$\begin{aligned} E_2^{p,q} &= \Lambda^q(\mathfrak{g}_{k-2}/\mathfrak{g}_{k-3})^{*(0,1)} \otimes H^p(M_{k-2}, \Theta_{M_{k-2}}) \\ &= \Lambda^q(\mathfrak{g}_{k-2}/\mathfrak{g}_{k-3})^{*(0,1)} \otimes H^p_{\bar{\partial}}((\mathfrak{g}/\mathfrak{g}_{k-2})^{1,0}), \end{aligned}$$

by the first inductive step.

Elements in $\Lambda^q(\mathfrak{g}_{k-2}/\mathfrak{g}_{k-3})^{*(0,1)}$ are pulled back to globally defined $(0, q)$ -forms on M_{k-3} . Elements of $H^p_{\bar{\partial}}((\mathfrak{g}/\mathfrak{g}_{k-2})^{1,0})$ yield globally defined forms on M_{k-3} and globally defined sections of $\pi_{k-2}^* \Theta_{M_{k-2}}$. Moreover, $d_2 = 0$, since d_2 is generated by $\bar{\partial}$. So the spectral sequence degenerates at E_2 . Thus

$$\begin{aligned} H^j(M_{k-3}, \pi_{k-2}^* \Theta_{M_{k-2}}) \\ = \bigoplus_{p+q=j} E_2^{p,q} = \bigoplus_{p+q=j} \Lambda^q(\mathfrak{g}_{k-2}/\mathfrak{g}_{k-3})^{*(0,1)} \otimes H^p_{\bar{\partial}}((\mathfrak{g}/\mathfrak{g}_{k-2})^{1,0}). \end{aligned} \tag{3.1}$$

The latter is the cohomology of the complex

$$\begin{array}{ccc} \bigoplus_{p+q=j} (\mathfrak{g}_{k-2}/\mathfrak{g}_{k-3})^{*(0,q)} \otimes ((\mathfrak{g}/\mathfrak{g}_{k-2})^{*(0,p)} \otimes (\mathfrak{g}/\mathfrak{g}_{k-2})^{1,0}) & & \\ \downarrow 0 & & \downarrow \bar{\partial} \\ \bigoplus_{p+q=j} (\mathfrak{g}_{k-2}/\mathfrak{g}_{k-3})^{*(0,q)} \otimes ((\mathfrak{g}/\mathfrak{g}_{k-2})^{*(0,p+1)} \otimes (\mathfrak{g}/\mathfrak{g}_{k-2})^{1,0}). & & \end{array}$$

Note that $\mathfrak{g}_{k-2}/\mathfrak{g}_{k-3}$ is in the center of $\mathfrak{g}/\mathfrak{g}_{k-3}$. If V is in $\mathfrak{g}/\mathfrak{g}_{k-2}$ and \bar{X}_j is in $\mathfrak{g}/\mathfrak{g}_{k-3}$, $[\bar{X}_j, V]_{\mathfrak{g}/\mathfrak{g}_{k-3}}$ is well defined with respect to the induced Lie bracket on the quotient algebra $\mathfrak{g}/\mathfrak{g}_{k-3}$. Recall that $\{\bar{\omega}_j : 1 \leq j \leq n_\ell\}$ forms a basis for $\mathfrak{g}/\mathfrak{g}_\ell$. Its dual basis is $\{\bar{X}_j : 1 \leq j \leq n_\ell\}$. Now we consider a linear map $\bar{\partial}'$ on as follows. For $\bar{\omega}$ in $(\mathfrak{g}/\mathfrak{g}_{k-3})^{*(0,j)}$ and V in $(\mathfrak{g}/\mathfrak{g}_{k-2})^{1,0}$, define

$$\bar{\partial}'(\bar{\omega} \otimes V) = \sum_{j=1}^{n-n_{k-3}} (-1)^j \bar{\omega} \wedge \bar{\omega}_j \otimes \pi_{k-2}([\bar{X}_j, V]_{\mathfrak{g}/\mathfrak{g}_{k-3}})^{1,0}. \tag{3.2}$$

It yields a linear map

$$\bar{\partial}' : (\mathfrak{g}/\mathfrak{g}_{k-3})^{*(0,j)} \otimes (\mathfrak{g}/\mathfrak{g}_{k-2})^{1,0} \rightarrow (\mathfrak{g}/\mathfrak{g}_{k-3})^{*(0,j+1)} \otimes (\mathfrak{g}/\mathfrak{g}_{k-2})^{1,0}. \tag{3.3}$$

Now, using the fact that J is an abelian complex structure, one could verify that $\bar{\partial}' \circ \bar{\partial}' = 0$. Moreover, if \bar{X}_j is dual to an element of $\mathfrak{g}_{k-2}/\mathfrak{g}_{k-3}$, then $[\bar{X}_j, V]_{\mathfrak{g}/\mathfrak{g}_{k-3}} = 0$. Therefore, the complex with $\bar{\partial}'$ and the complex $0 \otimes \bar{\partial}$ in the previous paragraph agree.

We denote by $H_{\bar{\partial}}^j((\mathfrak{g}/\mathfrak{g}_{k-3}, \mathfrak{g}/\mathfrak{g}_{k-2})^{1,0})$ the cohomology of the above $\bar{\partial}'$ -complex. (It is a sort of “relative” cohomology.)

Next, we can describe the $\bar{\partial}$ -complex on $\mathfrak{g}/\mathfrak{g}_{k-3}$ using the splitting

$$\mathfrak{g}/\mathfrak{g}_{k-3} = \mathfrak{g}/\mathfrak{g}_{k-2} \oplus \mathfrak{g}_{k-3}/\mathfrak{g}_{k-2} = \mathfrak{g}/\mathfrak{g}_{k-2} \oplus \mathfrak{t}_{k-2}.$$

It yields

$$\begin{array}{ccc} ((\mathfrak{g}/\mathfrak{g}_{k-3})^{*(0,j)} \otimes (\mathfrak{g}/\mathfrak{g}_{k-2})^{1,0}) & \oplus & ((\mathfrak{g}/\mathfrak{g}_{k-3})^{*(0,j)} \otimes \mathfrak{t}_{k-2}^{1,0}) \\ \bar{\partial}' \downarrow & \searrow \bar{\partial}'' & \downarrow 0 \\ ((\mathfrak{g}/\mathfrak{g}_{k-3})^{*(0,j+1)} \otimes (\mathfrak{g}/\mathfrak{g}_{k-2})^{1,0}) & \oplus & ((\mathfrak{g}/\mathfrak{g}_{k-3})^{*(0,j+1)} \otimes \mathfrak{t}_{k-2}^{1,0}), \end{array}$$

where $\bar{\partial}''$ is the “ $\mathfrak{t}_{k-2}^{1,0}$ -component” of $\bar{\partial}$ (in a similar sense as in (3.2)).

Now, using the $\bar{\partial}'$ -complex (3.3) and the “relative” cohomology $H_{\bar{\partial}}^j((\mathfrak{g}/\mathfrak{g}_{k-3}, \mathfrak{g}/\mathfrak{g}_{k-2})^{1,0})$, (3.1) becomes

$$H^j(M_{k-3}, \pi_{k-3}^* \Theta_{M_{k-2}}) \cong H_{\bar{\partial}}^j((\mathfrak{g}/\mathfrak{g}_{k-3}, \mathfrak{g}/\mathfrak{g}_{k-2})^{1,0}).$$

Consider now the exact sequence

$$0 \rightarrow (\mathfrak{g}_{k-2}/\mathfrak{g}_{k-3})^{1,0} \otimes \mathcal{O}_{M_{k-3}} \rightarrow \Theta_{M_{k-3}} \rightarrow \pi_{k-3}^* \Theta_{M_{k-2}} \rightarrow 0,$$

or (setting as usual $\mathfrak{t}_{k-2} := \mathfrak{g}_{k-2}/\mathfrak{g}_{k-3}$)

$$0 \rightarrow \mathfrak{t}_{k-2}^{1,0} \otimes \mathcal{O}_{M_{k-3}} \rightarrow \Theta_{M_{k-3}} \rightarrow \pi_{k-3}^* \Theta_{M_{k-2}} \rightarrow 0,$$

which induces the long exact sequence

$$\begin{aligned} \dots &\rightarrow \mathfrak{t}_{k-2}^{1,0} \otimes H^j(M_{k-3}, \mathcal{O}_{M_{k-3}}) \rightarrow H^j(M_{k-3}, \Theta_{M_{k-3}}) \rightarrow H^j(M_{k-3}, \pi_{k-3}^* \Theta_{M_{k-2}}) \\ &\xrightarrow{\delta_j} \mathfrak{t}_{k-2}^{1,0} \otimes H^{j+1}(M_{k-3}, \mathcal{O}_{M_{k-3}}) \rightarrow \dots \end{aligned}$$

By the previous results this sequence can be written as

$$\begin{aligned} \dots &\rightarrow (\mathfrak{g}/\mathfrak{g}_{k-3})^{*(0,j)} \otimes \mathfrak{t}_{k-2}^{1,0} \rightarrow H^j(M_{k-3}, \Theta_{M_{k-3}}) \rightarrow H_{\bar{\partial}}^j((\mathfrak{g}/\mathfrak{g}_{k-3}, \mathfrak{g}/\mathfrak{g}_{k-2})^{1,0}) \\ &\xrightarrow{\delta_j} (\mathfrak{g}/\mathfrak{g}_{k-3})^{*(0,j+1)} \otimes \mathfrak{t}_{k-2}^{1,0} \rightarrow \dots \end{aligned}$$

To compute the coboundary map δ_j , we chase diagram and find that it is precisely the map $\bar{\partial}''$ on the “relative” cohomology $H_{\bar{\partial}}^j((\mathfrak{g}/\mathfrak{g}_{k-3}, \mathfrak{g}/\mathfrak{g}_{k-2})^{1,0})$. Given the long exact sequence we have

$$H^j(M_{k-3}, \Theta_{M_{k-3}}) \cong \ker \delta_j \oplus \frac{(\mathfrak{g}/\mathfrak{g}_{k-3})^{*(0,j)} \otimes \mathfrak{t}_{k-2}^{1,0}}{\delta_{j-1}(H_{\bar{\partial}}^{j-1}((\mathfrak{g}/\mathfrak{g}_{k-3}, \mathfrak{g}/\mathfrak{g}_{k-2})^{1,0}))}.$$

Let us compare this with $H_{\bar{\partial}}^j((\mathfrak{g}/\mathfrak{g}_{k-3})^{1,0})$, which is the cohomology of the complex determined by $\bar{\partial} = \bar{\partial}' + \bar{\partial}''$. Let $[b + t] \in H_{\bar{\partial}}^j((\mathfrak{g}/\mathfrak{g}_{k-3})^{1,0})$, with $b \in (\mathfrak{g}/\mathfrak{g}_{k-3})^{*(0,j)} \otimes$

$(\mathfrak{g}/\mathfrak{g}_{k-2})^{1,0}$ and $t \in (\mathfrak{g}/\mathfrak{g}_{k-3})^{*(0,j)} \otimes \mathfrak{t}_{k-2}^{1,0}$. Then $b \in \ker \bar{\partial} = \ker \bar{\partial}' \cap \ker \bar{\partial}''$. Moreover $b + t$ is cohomologous to $b' + t$ if and only if $b - b' \in \text{Im } \bar{\partial} = \text{Im } \bar{\partial}'$. Thus

$$\frac{\ker \bar{\partial}' \cap \ker \bar{\partial}''}{\text{Im } \bar{\partial}'}$$

determines the cohomology class of b . This is also the space $\ker \delta_j$.

Next, observe further that $b + t$ is cohomologous to $b + t'$ if and only if $t - t' \in \text{Im } \bar{\partial} = \text{Im } \bar{\partial}''$, i.e. $t - t' \in \bar{\partial}''(b'')$. Since b'' is mapped by $\bar{\partial}$ into $(\mathfrak{g}/\mathfrak{g}_{k-3})^{*(0,j)} \otimes \mathfrak{t}_{k-2}^{1,0}$, $\bar{\partial}' b'' = 0$, so $b'' \in \ker \bar{\partial}'$. Hence $t - t' \in \bar{\partial}''(\ker \bar{\partial}') = \delta_{j-1}(H_{\bar{\partial}}^{j-1}((\mathfrak{g}/\mathfrak{g}_{k-3}, \mathfrak{g}/\mathfrak{g}_{k-2})^{1,0}))$. Therefore

$$H_{\bar{\partial}}^j((\mathfrak{g}/\mathfrak{g}_{k-3})^{1,0}) = \ker \delta_j \oplus \frac{(\mathfrak{g}/\mathfrak{g}_{k-3})^{*(0,j)} \otimes \mathfrak{t}_{k-2}^{1,0}}{\delta_{j-1}(H_{\bar{\partial}}^{j-1}((\mathfrak{g}/\mathfrak{g}_{k-3}, \mathfrak{g}/\mathfrak{g}_{k-2})^{1,0}))}$$

It is therefore isomorphic to $H^j(M_{k-3}, \Theta_{M_{k-3}})$ as claimed.

With a shift of indices, it established an inductive process to complete the proof of the theorem. □

4. Deformation Theory

We will show that one can find harmonic representatives in the Dolbeault cohomology groups. To this goal, we introduce an invariant Hermitian metric on M . We choose such a metric so that if $\{X_1, \bar{X}_1, \dots, X_n, \bar{X}_n\}$ is the real basis of \mathfrak{g} constructed in Sec. 2, $\{Y_1, JY_1, \dots, Y_n, JY_n\}$ forms a Hermitian frame, where we set $Y_i := \frac{1}{2}(X_i + \bar{X}_i)$. We use the resulting inner product on $\mathfrak{g}^{*(0,k)} \otimes \mathfrak{g}^{1,0}$ to define the orthogonal complement of $\text{Im } \bar{\partial}_{k-1}$ in $\ker \bar{\partial}_k$. Denote this space by $\text{Im}^\perp \bar{\partial}_{k-1}$. Then, with the same proof as in [7, Theorem 3], we find that the harmonic theory is reduced to finite-dimensional linear algebra.

Theorem 4.1. *The space $\text{Im}^\perp \bar{\partial}_{k-1}$ is a space of harmonic representatives for the Dolbeault cohomology $H^k(M, \Theta_M)$ on the compact complex manifold M . In addition, let $\mu \in \mathfrak{g}^{*(0,k)} \otimes \mathfrak{g}^{1,0}$. Then $\bar{\partial}^* \mu$ with respect to the L_2 -norm on the compact manifold M is equal to $\bar{\partial}^* \mu$ with respect to the Hermitian inner product on the finite-dimensional vector spaces $\mathfrak{g}^{*(0,k)} \otimes \mathfrak{g}^{1,0}$.*

Next one can consider the Schouten–Nijenhuis bracket

$$\{\cdot, \cdot\} : H^1(X, \Theta_X) \times H^1(X, \Theta_X) \rightarrow H^2(X, \Theta_X).$$

Recall that it can be defined as follows. Let $\bar{w} \otimes V$ and $\bar{w}' \otimes V'$ be vector-valued $(0, 1)$ -forms representing elements in $H^1(X, \Theta_X)$. Then

$$\{\bar{w} \otimes V, \bar{w}' \otimes V'\} = \bar{w}' \wedge L_{V'} \bar{w} \otimes V + \bar{w} \wedge L_V \bar{w}' \otimes V' + \bar{w} \wedge \bar{w}' \otimes [V, V']$$

Using the fact that the complex structure is abelian, so $[V, V'] = 0$ for all $(1, 0)$ -vectors, one gets

$$\{\bar{\omega} \otimes V, \bar{\omega}' \otimes V'\} = \bar{\omega}' \wedge \iota_{V'} d\bar{\omega} \otimes V + \bar{\omega} \wedge \iota_V d\bar{\omega}' \otimes V'. \tag{4.1}$$

To construct deformations, we can now apply Kuranishi’s recursive formula like in [7]. We recall it for the sake of completeness.

Let $\{\beta_1, \dots, \beta_N\}$ be an orthonormal basis of the harmonic representatives of $H^1(M, \Theta_M)$. For any vector $\mathbf{t} = (t_1, \dots, t_N)$ in \mathbb{C}^N , let $\boldsymbol{\mu}(\mathbf{t}) = t_1\beta_1 + \dots + t_N\beta_N$ and set $\phi_1 = \boldsymbol{\mu}$. Then, one can define ϕ_r inductively for $r \geq 2$ as we will now recall.

Denote as usual by $\bar{\partial}^*$ the adjoint operator to the $\bar{\partial}$ -operator on M with respect to the Hermitian metric previously defined and by $\Delta = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ the Laplacian. Then we set

$$\phi_r(\mathbf{t}) = \frac{1}{2} \sum_{s=1}^{r-1} \bar{\partial}^* \mathcal{G}\{\phi_s(\mathbf{t}), \phi_{r-s}(\mathbf{t})\} = \frac{1}{2} \sum_{s=1}^{r-1} \mathcal{G}\bar{\partial}^* \{\phi_s(\mathbf{t}), \phi_{r-s}(\mathbf{t})\}, \tag{4.2}$$

where \mathcal{G} is the corresponding Green’s operator that inverts Δ on the orthogonal complement of the space of harmonic forms.

Consider the formal sum

$$\Phi(\mathbf{t}) = \sum_{r \geq 1} \phi_r. \tag{4.3}$$

Observe that Φ belongs to $\mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0}$ and can be regarded either as a map sending $\mathfrak{g}^{0,1}$ to $\mathfrak{g}^{1,0}$ or as one from $\mathfrak{g}^{*(1,0)}$ to $\mathfrak{g}^{*(0,1)}$.

Let $\{\gamma_1, \dots, \gamma_P\}$ be an orthonormal basis for the space of harmonic $(0, 2)$ -forms with values in Θ_M . Define $f_k(\mathbf{t})$ to be the L^2 -inner product $\langle\langle \{\Phi(\mathbf{t}), \Phi(\mathbf{t})\}, \gamma_k \rangle\rangle$. Kuranishi theory [6] asserts the existence of $\epsilon > 0$ such that

$$\{\mathbf{t} \in \mathbb{C}^N : |\mathbf{t}| < \epsilon, f_1(\mathbf{t}) = 0, \dots, f_P(\mathbf{t}) = 0\} \tag{4.4}$$

forms a locally complete family of deformations of M . We shall denote this set by Kur.

For each $\mathbf{t} \in \text{Kur}$, the associated sum $\Phi = \Phi(\mathbf{t})$ defines a family of complex structures J_Φ whose $(1, 0)$ -forms are given by $\omega - \Phi(\omega)$, $\omega \in \mathfrak{g}^{*(1,0)}$ and whose $(0, 1)$ -vectors are $\bar{X} + \Phi(\bar{X})$, $\bar{X} \in \mathfrak{g}^{0,1}$.

Consequently the integrability condition for a deformation Φ is, for any $\omega \in \mathfrak{g}^{*(1,0)}$ and $\bar{X}, \bar{Y} \in \mathfrak{g}^{0,1}$,

$$(d(\omega - \Phi(\omega)))(\bar{X} + \Phi(\bar{X}), \bar{Y} + \Phi(\bar{Y})) = 0. \tag{4.5}$$

To relate this condition to the Maurer–Cartan equation, one may check that

$$-(d(\omega - \Phi(\omega)))(\bar{X} + \Phi(\bar{X}), \bar{Y} + \Phi(\bar{Y})) = \omega \left(\left(\bar{\partial}\Phi + \frac{1}{2}\{\Phi, \Phi\} \right) (\bar{X}, \bar{Y}) \right). \tag{4.6}$$

Now, given the fact that the harmonic theory is reduced to a finite-dimensional program as noted in Theorem 4.1, a proof in [7] shows that every term in the power

series (4.3) lies in $\mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0}$. Thus

Theorem 4.2. *Let G be a nilpotent Lie group with co-compact subgroup Γ , and let J be an abelian invariant complex structure on $M = \Gamma \backslash G$. Then the deformations arising from J parameterized by (4.4) are all invariant complex structures.*

We can now find under which conditions J_{Φ} remains abelian. Recall that a complex structure is abelian if and only if the differential of a $(1, 0)$ form is of type $(1, 1)$. In other words, for any $\omega \in \mathfrak{g}^{*(1,0)}$ and $X, Y \in \mathfrak{g}^{1,0}$,

$$d(\omega - \Phi(\omega))(X + \Phi(X), Y + \Phi(Y)) = 0. \tag{4.7}$$

If one extends the Schouten–Nijenhuis bracket $\{\cdot, \cdot\}$ to the exterior algebra by anti-derivative as seen in [9] and make use of the assumption that J is abelian, a short computation shows that

$$d(\omega - \Phi(\omega))(X + \Phi(X), Y + \Phi(Y)) = \{\bar{\Phi}, \omega - \Phi(\omega)\}(X, Y). \tag{4.8}$$

Note that $\Phi(\omega)$ is in $\mathfrak{g}^{*(0,1)}$, so $\{\bar{\Phi}, \Phi(\omega)\}$ is in $\mathfrak{g}^{*(0,2)}$. Note further that if α is in $\mathfrak{g}^{*(1,0)}$ and \bar{V} is in $\mathfrak{g}^{0,1}$, then

$$\{\alpha \otimes \bar{V}, \omega\} = \alpha \wedge \{\bar{V}, \omega\} = \alpha \wedge \iota_{\bar{V}} d\omega. \tag{4.9}$$

Since the complex structure is abelian, $d\omega$ is type $(1, 1)$. Therefore, $\iota_{\bar{V}} d\omega$ is in $\mathfrak{g}^{*(1,0)}$. Therefore, $\{\bar{\Phi}, \omega - \Phi(\omega)\}$ is in $\mathfrak{g}^{*(2,0)} \otimes \mathfrak{g}^{1,0}$. It follows from Eq. (4.8) above that the deformed complex structure J_{Φ} is abelian if and only if

$$\{\bar{\Phi}, \omega - \Phi(\omega)\} = 0 \tag{4.10}$$

for all $\omega \in \mathfrak{g}^{*(1,0)}$.

Theorem 4.3. *Φ defines an abelian deformation if and only if it is integrable and*

$$\{\bar{\Phi}, \omega - \Phi(\omega)\} = 0 \quad \text{for any } \omega \in \mathfrak{g}^{*(1,0)}.$$

Infinitesimally, suppose $\Phi = t\mu + t^2\phi_2 + t^3\phi_3 + \dots$. Then looking at the degree one terms, the integrability condition implies that $\bar{\partial}\mu = 0$. Computing the second order term in Eq. (4.10) yields

$$\{\mu, \bar{\omega}\} = 0 \tag{4.11}$$

for any $\bar{\omega}$ in $\mathfrak{g}^{*(0,1)}$. It follows that if ϕ_j are constructed through the Kuranishi recursive formula, then $\phi_j = 0$ for all $j \geq 2$. Conversely, if μ is $\bar{\partial}$ -closed and satisfies the above condition, it is integrable to an abelian complex structure. Therefore, we have the following result.

Proposition 4.4. *A parameter $\mu \in H^1(M, \Theta_M)$ defines an integrable infinitesimal abelian deformation if and only if $\bar{\partial}\mu = 0$ and*

$$\{\mu, \bar{\omega}\} = 0 \quad \text{for any } \bar{\omega} \in \mathfrak{g}^{*(0,1)}.$$

Furthermore, we may consider μ as an element in $\mathfrak{g}^{1,0} \oplus \mathfrak{g}^{*(0,1)}$. With the Schouten bracket, the space $\mathfrak{g}^{1,0} \oplus \mathfrak{g}^{*(0,1)}$ becomes a Lie algebra. Since the complex structure is abelian, $\{\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}\} = 0$. By definition, $\{\mathfrak{g}^{*(0,1)}, \mathfrak{g}^{*(0,1)}\} = 0$. Therefore, $\{\mu, \bar{\omega}\} = 0$ for any $\omega \in \mathfrak{g}^{*(0,1)}$ if and only if μ is in the kernel of the adjoint map.

Corollary 4.5. *A parameter $\mu \in H^1(M, \Theta_M)$ defines an integrable infinitesimal abelian deformation if and only if $\bar{\partial}\mu = 0$ and μ is in the center of the Lie algebra $\mathfrak{g}^{1,0} \oplus \mathfrak{g}^{*(0,1)}$.*

5. Six-Dimensional Examples

We recall that given a complex structure J on a $2n$ -dimensional nilpotent Lie algebra \mathfrak{g} , one may consider the ascending series $\{\mathfrak{g}_l^J\}$ defined inductively by $\mathfrak{g}_0^J = \{0\}$ and

$$\mathfrak{g}_l^J = \{X \in \mathfrak{g} : [X, \mathfrak{g}] \subseteq \mathfrak{g}_{l-1}^J, [JX, \mathfrak{g}] \subseteq \mathfrak{g}_{l-1}^J\}, \quad l \geq 1.$$

The complex structure is said to be nilpotent if the series satisfies $\mathfrak{g}_k^J = \mathfrak{g}$, for some positive integer k [5]. Apparently, an abelian complex structure is nilpotent, with $\mathfrak{g}_\ell^J = \mathfrak{g}_\ell$, for any $\ell \geq 0$.

By [12, Corollary 2.7] if G is a 6-dimensional nilpotent Lie group admitting an invariant complex structure, then all invariant complex structures are either nilpotent or non-nilpotent altogether. Thus, combining this result with Theorem 4.2, one has the following observation.

Corollary 5.1. *Let G be a 6-dimensional nilpotent Lie group with co-compact subgroup Γ , and let J be an abelian invariant complex structure on $M = \Gamma \backslash G$. Then the deformations arising from J parameterized by (4.4) are all invariant nilpotent complex structures.*

Abelian complex structures on 2-step nilmanifolds with dimension six are already studied in [7]. Using the classification obtained in [11, 12], in dimension 6, there are only two k -step nilpotent Lie algebras with $k > 2$. Their respective structure equations are as follow.

$$\begin{aligned} \mathfrak{h}_9 : [e_1, e_2] &= e_3, [e_1, e_3] = [e_2, e_4] = e_6, \\ \mathfrak{h}_{15} : [e_1, e_2] &= -e_4, [e_1, e_3] = [e_2, e_4] = e_5, [e_1, e_4] = -[e_2, e_3] = -e_6. \end{aligned}$$

Both are 3-step nilpotent.

It is known that any complex structure on the first one \mathfrak{h}_9 is abelian [12, Theorem 2.9]. Thus, if H_9 is the simply connected nilpotent Lie group with Lie algebra \mathfrak{h}_9 and Γ is a co-compact subgroup Γ , the deformations arising from any invariant complex structure J on $\Gamma \backslash H_9$ parameterized by (4.4) are all invariant abelian complex structures.

We can actually verify directly that, if we start from the abelian complex structure J on \mathfrak{h}_9 with

$$Je_1 = e_2, \quad Je_3 = e_4, \quad Je_5 = e_6,$$

any Kuranishi deformation is still abelian. More explicitly,

$$X_1 = e_5 - ie_6, \quad X_2 = e_3 - ie_4, \quad X_3 = e_1 - ie_2$$

is a basis of left invariant $(1, 0)$ fields. Moreover

$$\bar{\omega}^1 = e^5 - ie^6, \quad \bar{\omega}^2 = e^3 - ie^4, \quad \bar{\omega}^3 = e^1 - ie^2,$$

where e^i are the dual of e_j gives a basis of $(0, 1)$ -forms. An orthonormal basis of harmonic representatives for $H_{\mathfrak{g}}^1(\mathfrak{h}_9^{1,0})$ is given by

$$\beta_1 = \bar{\omega}^1 \otimes X_1, \quad \beta_2 = \frac{1}{\sqrt{2}}(\bar{\omega}^2 \otimes X_2 - \bar{\omega}^3 \otimes X_3), \quad \beta_3 = \frac{1}{\sqrt{2}}(\bar{\omega}^2 \otimes X_1 - \bar{\omega}^3 \otimes X_2).$$

Now if $\mu = a_1\beta_1 + a_2\beta_2 + a_3\beta_3$ one can check that $\{\mu, \bar{\omega}^i\} = 0$ for any $i = 1, 2, 3$, which shows directly that any small Kuranishi deformation of \mathfrak{h}_9 arising from J is abelian.

We also verify directly that Kuranishi deformations of abelian complex structures on \mathfrak{h}_{15} are not necessarily abelian. Let us start from the abelian complex structure J on \mathfrak{h}_{15} with

$$Je_1 = e_2, \quad Je_3 = e_4, \quad Je_5 = e_6.$$

Like in the previous case,

$$X_1 = e_5 - ie_6, \quad X_2 = e_3 - ie_4, \quad X_3 = e_1 - ie_2, \\ \bar{\omega}^1 = e^5 - ie^6, \quad \bar{\omega}^2 = e^3 - ie^4, \quad \bar{\omega}^3 = e^1 - ie^2,$$

are basis of left invariant $(1, 0)$ fields and $(0, 1)$ -forms respectively, where e^i are the dual of e_j . An orthonormal basis of harmonic representatives for $H_{\mathfrak{g}}^1(\mathfrak{h}_{15}^{1,0})$ is given by

$$\beta_1 = \bar{\omega}^1 \otimes X_1, \quad \beta_2 = \frac{1}{\sqrt{5}}(\bar{\omega}^2 \otimes X_1 - 2\bar{\omega}^3 \otimes X_2), \quad \beta_3 = \bar{\omega}^3 \otimes X_1, \\ \beta_4 = \bar{\omega}^1 \otimes X_2, \quad \beta_5 = \bar{\omega}^2 \otimes X_2.$$

Setting $\mu = \sum_{\ell=1}^5 a_{\ell}\beta_{\ell}$, we have $\{\mu, \bar{\omega}^{\ell}\} = 0$ for any $\ell = 1, 2, 3$ if and only if $a_4 = a_5 = 0$. Hence any such μ with non-vanishing a_4 or a_5 parametrizes deformations of J which are invariant and nilpotent, but not abelian.

6. Higher Dimension Example

In higher dimensions, deformations of abelian complex structures are not necessarily nilpotent. Here is an example of an abelian complex structure on a ten-dimensional 3-step nilpotent Lie algebra \mathfrak{n} which deforms into non-nilpotent complex structures.

The nilpotent Lie algebra \mathfrak{n} has structure equations:

$$de^1 = de^2 = de^3 = de^7 = 0, \quad de^4 = -e^{12} + e^{13} + e^{27}, \quad de^5 = -e^{12} - e^{17} + e^{23}, \\ de^6 = -e^{14} - e^{15} - e^{25} + e^{24} - e^{19} + e^{28} - 2e^{45} + e^{48} + e^{59} - e^{49} + e^{58} - e^{89}, \\ de^8 = -e^{17} + e^{23} - e^{13} - e^{27}, \quad de^9 = 2e^{12} + e^{17} - e^{23} - e^{13} - e^{27}, \quad de^{10} = 0.$$

Consider the family of complex structure $J_{s,t}$ on \mathfrak{n} such that

$$\begin{aligned} J_{s,t}e_1 &= e_2, & J_{s,t}e_4 &= e_5, & J_{s,t}e_8 &= e_9, \\ J_{s,t}e_3 &= te_6 + se_7, & J_{s,t}e_{10} &= -se_6 - te_7, \\ J_{s,t}e_6 &= \frac{1}{t^2 - s^2}(-te_3 - se_{10}), & J_{s,t}e_7 &= \frac{1}{t^2 - s^2}(se_3 + te_{10}), \end{aligned}$$

with s, t real parameters such that $t^2 \neq s^2$. To check integrability, we choose the $(1, 0)$ -forms as follows.

$$\begin{aligned} \omega^1 &= e^1 + ie^2, & \omega^2 &= e^4 + ie^5, & \omega^3 &= e^8 + ie^9, \\ \omega^4 &= e^3 + i(te^6 + se^7), & \omega^5 &= e^{10} - i(se^6 + te^7). \end{aligned}$$

It suffices to show that $d\omega^j$ is of type $(1, 1) + (2, 0)$, i.e. it is generated by the ideal of type $(1, 0)$ -forms. It is a long, yet straight forward exercise. We do not display all formula here.

For $t = 0, s = 1$, one has an abelian complex structure. Note that the center of \mathfrak{n} is spanned by e_6 and e_{10} . If a complex structure J is nilpotent, then \mathfrak{g}_1^J is non-trivial, J -invariant and contained in the center of \mathfrak{g} . Given the dimension of the center, it is possible only if the center of \mathfrak{g} is equal to \mathfrak{g}_1^J . In particular, it is J -invariant. Note that the center is not preserved by $J_{s,t}$ if $t \neq 0$. Thus a generic $J_{s,t}$ is not nilpotent.

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