Weak Mirror Symmetry of Nilmanifolds

R. Cleyton * J. Lauret † Y. S. Poon ‡

Presented on the occasion of Prof. A. Fujiki's 60th birthday
by Y. S. Poon

1 Introduction

It is well known that given a complex structure J on a smooth manifold M there is a differential Gerstenhaber algebra DGA(M, J). Such objects first appeared as Gerstenhaber investigated the deformations of rings generalizing the deformation theory of compact complex manifolds by Kodaira and Spencer [14]. Part of this algebra turns out to be realized by a deformation theory of "generalized complex structures" [16]. The full algebra is a central object in homological mirror symmetry theory [2].

Since Moser's Theorem, symplectic structures are known to have no non-trivial deformation. Nonetheless, associated to every symplectic structure ω on a manifold

*Address: Institute für Mathematatik, Humboldt Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany. E-mail:cleyton@mathematik.hu-berlin.de

†Address: FaMAF and CIEM, Universidad Nacional de Córdoba, 5000 Córdoba, Argentina. E-mail:lauret@mate.uncor.edu

‡Address: Department of Mathematics, University of California at Riverside, Riverside CA 92521, U.S.A.. E-mail: ypoon@ucr.edu

N, there is a differential Gerstenhaber algebra DGA(N, ω). Again part of this algebra is realized by deformation theory of generalized complex structures, which include symplectic structures as examples.

Due to Merkulov, the structures (M, J) and (N, ω) are said to form a weak mirror pair if DGA(M, J) and DGA(N, ω) are quasi-isomorphic [20]. With a focus on nilmanifolds, in this lecture I present the background of this subject, some past results, and a preliminary report on a complete description of weak mirror pairs of six-dimensional nilmanifolds. The work on six-dimension is due to my collaboration with R. Cleyton in the past few years, and recently with J. Lauret.

Our initial investigation on the differential Gerstenhaber algebra theory is inspired by homological mirror symmetry theory as developed in [2], [19], [20], [27]. Our techniques of constructing mirror pairs are inspired by the SYZ conjecture's utilization of special Lagrangian torus [25] and its recent development due to Arinkin and Polishchuk [1], subsequently Ben-Bassat [4] [5].

2 Nilmanifolds

A manifold M is a nilmanifold if it is the quotient space $\Gamma\backslash H$ where H is a simply connected nilpotent Lie group and Γ is a co-compact lattice in H.

We shall focus on invariant complex structures and symplectic structures on such manifolds. A complex structure, a symplectic structure or any tensorial object on M is said to be invariant if it is lifted to a left-invariant tensor to the Lie group H via the natural quotient map.

An obvious example of nilmanifolds with invariant complex or symplectic structures is an even-dimensional torus. A well known non-trivial example is the Kodaira-Thurston surface [26], a two-dimensional compact complex surface. The Lie algebra $\mathfrak{h}$ of the covering space H is the direct sum of a real three-dimensional Heisenberg algebra and the one-dimensional algebra. If we use $\langle e_1, e_2, e_3, e_4\rangle$ to represent the real linear span of the vectors e_1, e_2, e_3, e_4, then the sole non-zero structure equation of the algebra $\mathfrak{h}$ is

$$[e_1, e_2] = -e_3.$$

In subsequent computation, we shall use upper indices to represent dual basis, and use the Chevalley-Eilenberg (a.k.a. C-E) differential d on the dual of a Lie algebra to present the structure equations. For the algebra $\mathfrak{h}$ in question, the sole non-trivial structure equation is

$$de^3 = e^1 \wedge e^2.$$

We shall also use e^{ij} to represent $e^i \wedge e^j$. In addition, to summarize the structure equations, we shall list $de^1, \ldots, de^4$ in an array. For example, if the structure equations of an algebra $\mathfrak{k}$ are $de^1 = 0$, $de^2 = 0$, $de^3 = e^{12}$ and $de^4 = e^{ij} + e^{kl}$ for some i, j, k, l, we represent the algebra $\mathfrak{k}$ by

$$\mathfrak{k} = (0, 0, 12, ij + kl). \tag{1}$$

For instance, the algebra of the covering space over the Kodaira-Thurston surface is $(0, 0, 12, 0)$.

The Kodaira-Thurston surface is made famous by Kodaira's complex structure J and Thurston's non-Kählerian type-(1,1) symplectic structure ω. In terms of the frame $\langle e_1, e_2, e_3, e_4\rangle$ and its dual, these structures are given as

$$Je_1 = e_2, \quad Je_2 = -e_1, \quad Je_3 = e_4, \quad Je_4 = -e_1. \tag{2}$$

$$\omega = e^1 \wedge e^3 + e^2 \wedge e^4 = \mathrm{Re}(e^1 + ie^2) \wedge (e^3 - ie^4). \tag{3}$$

Although the symplectic structure ω is not Kählerian, it is pseudo-Kählerian with signature $(2, 2)$.

The study of invariant complex structures, symplectic structures and pseudo-Kähler structures on nilmanifolds of at most six-dimension has seen some progress in the past ten years. The two-dimensional case is trivial as the algebra has to be abelian. In dimension four, other than the abelian algebra, the algebra $(0, 0, 12, 0)$ is the only one admitting invariant complex structures [24]. The algebras $(0, 0, 12, 0)$ and $(0, 0, 12, 13)$ are the only two admitting non-trivial symplectic structure. Subsequently, $(0, 0, 12, 0)$ is the only non-abelian four-dimensional algebra admitting an invariant pseudo-Kähler structures.

A classification theory in six-dimensional is much more involved. The classification of nilpotent algebras admitting symplectic and complex structures are done by Goze-Khakimdjanov [17] and Salamon [24] respectively. The classification of nilpotent algebras admitting pseudo-Kähler structures is completed by Cordero et. al. [11]. Readers are cautioned that classification of an algebra admitting a certain structure is different from classification of the structures as an algebra may admit non-trivial family of inequivalent complex or symplectic structures.

3 Differential Gerstenhaber Algebras

Let R be a ring with unit and let C be an R-algebra. Let $\mathfrak{a} = \oplus_{n\in\mathbb{Z}}\mathfrak{a}^n$ be a graded algebra over C. $\mathfrak{a}$ is a *differential Gerstenhaber algebra* (a.k.a. DGA) if there is an associative product $\wedge$, a graded commutative product $[-,-]$, and a differential d of degree $+1$ satisfying the following conditions.

- $(\mathfrak{a}, \wedge, d)$ is a graded differential associative algebra;
- $(\mathfrak{a}, [-,-], d)$ is an odd differential graded Lie algebra;
- The associative product $\wedge$ and the odd graded Lie bracket $[-,-]$ satisfy an odd distributive rule. Namely,

$$[a, b \wedge c] = [a, b] \wedge c + (-1)^{(\deg a+1)\deg b} b \wedge [a, c]. \tag{4}$$

The cohomology of a DGA with respect to the differential d will inherit the structure of a Gerstenhaber algebra. Two differential Gerstenhaber algebras are said to be quasi-isomorphic if there is a homomorphism from one to another so that the induced homomorphism between the cohomologies is an isomorphism. We are only interested in the DGAs associated to symplectic structures or complex structures.

Let $\mathfrak{h}$ be a Lie algebra over $\mathbb{R}$. The exterior algebra of the dual $\mathfrak{h}^*$ with the C-E differential d forms a differential graded associative algebra over $\mathbb{R}$.

Suppose that ω is a symplectic form. Through contraction it becomes a real linear skew-symmetric map $\omega : \mathfrak{h} \to \mathfrak{h}^*$. Define a bracket $[-,-]_\omega$ on $\mathfrak{h}^*$ by

$$[\alpha, \beta]_\omega := \omega[\omega^{-1}\alpha, \omega^{-1}\beta]. \tag{5}$$

By construction, the map ω is an Lie algebra isomorphism. The fact that $d\omega = 0$ ensures that when we use the axioms of DGA to extend the Lie bracket from $\mathfrak{h}^*$ naturally to its exterior algebra, we have a DGA. The differential Gerstenhaber algebra $(\wedge^\bullet \mathfrak{h}^*, [-,-]_\omega, \wedge, d)$ after complexification is denoted by $\mathrm{DGA}(\mathfrak{h}, \omega)$.

Suppose J is an *integrable complex structure* on a real Lie algebra $\mathfrak{h}$. Then the $\pm i$ eigenspaces $\mathfrak{h}^{(1,0)}$ and $\mathfrak{h}^{(0,1)}$ are complex Lie subalgebras of the complexified algebra $\mathfrak{h}_\mathbb{C}$. Let $\mathfrak{h}^{*(1,0)}$ and $\mathfrak{h}^{*(0,1)}$ be their respective dual spaces. Let $\mathfrak{f}$ be the exterior algebra generated by $\mathfrak{f}^1 := \mathfrak{h}^{(1,0)} \oplus \mathfrak{h}^{*(0,1)}$. The integrability of J implies that $\mathfrak{f}^1$ is closed under the *Courant bracket*

$$[A + \alpha \bullet B + \beta] := [A, B] + \iota_A d\beta - \iota_B d\alpha, \tag{6}$$

where A, B are in $\mathfrak{h}^{(1,0)}$ and α, β are in $\mathfrak{h}^{*(0,1)}$. A similar construction holds for the conjugate $\bar{\mathfrak{f}}$, generated by $\bar{\mathfrak{f}}^1 := \mathfrak{h}^{(0,1)} \oplus \mathfrak{h}^{*(1,0)}$.

Since $(\mathfrak{h}, [-,-])$ is a Lie algebra, the Chevalley-Eilenberg differential d is a differential on the exterior algebra $\wedge\mathfrak{h}^*$. The natural pairing on $(\mathfrak{h} \oplus \mathfrak{h}^*)_\mathbb{C}$, induces a complex linear isomorphism $(\mathfrak{f}^1)^* \cong \bar{\mathfrak{f}}^1$. Therefore, the C-E differential of the Lie algebra $\bar{\mathfrak{f}}^1$ is a map from $\mathfrak{f}^1$ to $\wedge^2\mathfrak{f}^1$. Denote this operator by $\overline{\partial}$. Similarly, we denote the C-E differential of $\mathfrak{f}^1$ by ∂. These differentials are canonically extended to differentials on the exterior algebras of $\mathfrak{f}^1$ and $\bar{\mathfrak{f}}^1$ respectively. Then the quadruples $(\mathfrak{f}, [-,-], \wedge, \overline{\partial})$ and $(\bar{\mathfrak{f}}, [-,-], \wedge, \partial)$ are differential Gerstenhaber algebras.

For a given Lie algebra $\mathfrak{h}$ and a choice of invariant complex structure J, we denote the differential Gerstenhaber algebra $(\mathfrak{f}, [-,-], \wedge, \overline{\partial})$ by $\mathrm{DGA}(\mathfrak{h}, J)$.

Definition 3.1 *A complex structure* $(\mathfrak{h}, J)$ *and a symplectic structure* $(\mathfrak{k}, \omega)$ *form a weak mirror pair if* $\mathrm{DGA}(\mathfrak{h}, J)$ *and* $\mathrm{DGA}(\mathfrak{k}, \omega)$ *are quasi-isomorphic. Quasi-isomorphism between* DGA*s are denoted by* $\approx$.

Remark: Isomorphism between Lie algebras is denoted by $\cong$.

We shall further narrow our consideration of weak mirror symmetry to pseudo-Kähler structures.

Definition 3.2 *A pair of pseudo-Kähler structures* $(\mathfrak{h}, J, \omega)$ *and* $(\widehat{\mathfrak{h}}, \widehat{J}, \widehat{\omega})$ *forms a weak mirror pair if* $\mathrm{DGA}(\mathfrak{h}, J) \approx \mathrm{DGA}(\widehat{\mathfrak{h}}, \widehat{\omega})$ *and* $\mathrm{DGA}(\mathfrak{h}, \omega) \approx \mathrm{DGA}(\widehat{\mathfrak{h}}, \widehat{J})$.

In the above construction of DGAs and the development of concepts of weak mirror pairs, we focus on invariant structures on Lie algebras. A similar construction takes place on manifolds with complex structures and symplectic structures. Suppose that $N = \Gamma' \backslash K$ is a nilmanifold with an invariant symplectic structure ω and $M = \Gamma \backslash H$ is another nilmanifold with an invariant complex structure J. There are corresponding construction of differential Gerstenhaber algebras $\mathrm{DGA}(\Gamma' \backslash K, \omega)$ and $\mathrm{DGA}(\Gamma \backslash H, J)$. Then (N, ω) and (M, J) form a mirror pair if and only if $\mathrm{DGA}(\Gamma' \backslash K, \omega) \approx \mathrm{DGA}(\Gamma \backslash H, J)$.

4 Four-dimensional Weak Mirror Pairs

In four dimension, the only non-abelian nilpotent algebra admitting an invariant pseudo-Kähler structure is $\mathfrak{h} = (0, 0, 12, 0)$. The Lie group of this algebra covers the Kodaira-Thurston surface M. As a complex manifold, it could be realized as an elliptic fibration over an elliptic curve [3]. Its classical deformation theory was described by Borcea [6]. In terms of the invariant frame $\{e_1, e_2, e_3, e_4\}$, the complex structure J is simply

$$Je_1 = e_2, \quad Je_2 = -e_1, \quad Je_3 = e_4, \quad Je_4 = -e_3$$

with $\langle e_3, e_4 \rangle$ determines the real vertical tangent space with respect to the classical elliptic fibration.

As a complex manifold, the Kodaira-Thurston surface has a generalization to elliptic fibration over high dimension torus. Making use of the elliptic fibration, one may apply a spectral sequence argument to prove that the complex structure of a Kodaira surface (M, J) is quasi-isomorphic to its invariant counterparts: $(\mathfrak{h}, J)$. i.e. $\mathrm{DGA}(M, J) \approx \mathrm{DGA}(\mathfrak{h}, J)$ [15] [22].

On the other hand, for any real numbers (u_1, v_1, u_2, v_2) with $\triangle := u_1^2 + v_1^2 - u_2^2 - v_2^2$, consider the 2-form

$$\omega := u_1(e^{13} - e^{24}) + v_1(e^{14} + e^{23}) + u_2(e^{13} + e^{24}) + v_2(e^{14} - e^{23}).$$

It is a closed 2-form. It is non-degenerate if and only if $\triangle \neq 0$. In this case, ω is a symplectic form. Due to Nomizu's classical result on deRham cohomology of nilmanifolds [21], $\mathrm{DGA}(M, \omega) \approx \mathrm{DGA}(\mathfrak{h}, \omega)$.

An explicit construction finally establishes that $\mathrm{DGA}(\mathfrak{h}, J) \approx \mathrm{DGA}(\mathfrak{h}, \omega)$ [22]. Since the above family contains Thurston's symplectic structure [26], one may consider the Kodaira complex structure and the Thurston symplectic structure as a mirror pair sharing the same background manifold.

5 Six-dimensional Weak Mirror Pairs

The development over Kodaira surfaces inspires our investigation in higher dimension, but the development requires substantial amount of work. Here I outline our process. A complete development of the concerned concepts, results and proofs appear elsewhere [12] [13].

Again, due to a classical result of Nomizu [21] on the deRham cohomology of nilmanifolds, we have $\mathrm{DGA}(\Gamma'\backslash K, \omega) \approx \mathrm{DGA}(\mathfrak{k}, \omega)$. On the other hand, the cohomology of $\mathrm{DGA}(\Gamma\backslash H, J)$ is the Dolbeault cohomology of $\Gamma\backslash H$ with coefficients in the sheaf of sections of the holomorphic tangent bundle. Due to Rollenske and his predecessors' investigations [7] [8] [9] [10] [23], one could derive that there are many instances when $\mathrm{DGA}(\Gamma\backslash H, J) \approx \mathrm{DGA}(\mathfrak{h}, J)$. Therefore weak mirror symmetry between invariant symplectic and invariant complex structures on nilmanifolds is often reduced to a weak mirror symmetry between the invariant structures on the corresponding Lie algebras. Hence we focus our analysis on quasi-isomorphism on the algebra level.

However, unlike on the Kodaira surface our construction of mirror pairs is not based on an elliptic fibration. Instead, we adapt the concept of special Lagrangian spaces in the SYZ conjecture in our computation [18] [25]. Since we are dealing with linear objects, special Lagrangian fibration on a Lie algebra $\mathfrak{h}$ could be realized as semi-direct product of a Lie subalgebra $\mathfrak{g}$ and an abelian ideal V so long as the geometry are "compatible" with the Lie theoretic structures. For a full motivation and justification for the next definition, readers are referred to [12].

Definition 5.1 [12] *Let $\mathfrak{h}$ be a Lie algebra with a semi-direct product structure $\mathfrak{h} = \mathfrak{g} \ltimes_\rho V$, where $\rho : \mathfrak{g} \to V$ is a representation of the algebra $\mathfrak{g}$ and $\dim \mathfrak{g} = \dim V$. Let (ω, J) be a pseudo-Kähler structure on $\mathfrak{h}$. Then $\mathfrak{h}$ and (ω, J) are said to be special Lagrangian if $J\mathfrak{g} = V$, $JV = \mathfrak{g}$ and $\omega|_{\mathfrak{g}} = 0 = \omega|_V$.*

Following suggestion of the SYZ-conjecture [25], we construct dual semi-direct product of $\mathfrak{h}$, namely

$$\widehat{\mathfrak{h}} := \mathfrak{g} \ltimes_{\rho^*} V^* \tag{7}$$

where $\rho^* : \mathfrak{g} \to V^*$ is the dual representation of ρ. Based on a combination of Lie theoretic computations, it is possible to determine the isomorphism classes of all six-dimensional nilpotent algebras underlying a semi-direct product structure. Then the dual semi-direct products could be subsequently identified.

Adding that the algebra should carry a pseudo-Kähler structure narrows the range of algebras, especially when the semi-direct structure is supposed to be special Lagrangian. In particular, we derive a list of candidates of semi-direct nilpotent algebras with complex structures $(\mathfrak{h}, J)$ and their dual semi-direct product with symplectic structures $(\widehat{\mathfrak{h}}, \omega)$ [12].

$(\mathfrak{h}, J)$	$\mathfrak{h}_1$	$\mathfrak{h}_4$	$\mathfrak{h}_7$	$\mathfrak{h}_8$	$\mathfrak{h}_9$	$\mathfrak{h}_{10}$	$\mathfrak{h}_{11}$
$(\widehat{\mathfrak{h}}, \omega)$	$\mathfrak{h}_1$	$\mathfrak{h}_7$	$\mathfrak{h}_4$	$\mathfrak{h}_8$	$\mathfrak{h}_9$	$\mathfrak{h}_{10}$	$\mathfrak{h}_{11}$

(8)

where the algebras are given as below.

$$\mathfrak{h}_1 = (0,0,0,0,0,0), \quad \mathfrak{h}_4 = (0,0,0,0,12,14+23) \quad \mathfrak{h}_7 = (0,0,0,12,13,23),$$
$$\mathfrak{h}_8 = (0,0,0,0,0,12), \quad \mathfrak{h}_9 = (0,0,0,0,12,14+25),$$
$$\mathfrak{h}_{10} = (0,0,0,12,13,14), \quad \mathfrak{h}_{11} = (0,0,0,12,13,14+23).$$

The name $\mathfrak{h}_k$ is due to a classification in [9].

Given this list, we construct special Lagrangian structures (J, ω) on the respective algebras as semi-direct product $\mathfrak{h} = \mathfrak{g} \ltimes_\rho V$ in [12].

Modeled on a construction in [1] [4] [5], we construct a complex structure $\widehat{J}$ and a symplectic structure $\widehat{\omega}$ on the dual semi-direct product $\widehat{\mathfrak{h}}$. Specifically, we treat ω as an isomorphism from $\mathfrak{g} \oplus V$ to $V^* \oplus \mathfrak{g}^*$. When $A \in \mathfrak{g}$ and $\alpha \in V^*$, we have

$$\widehat{J}(A) := \omega(A), \quad \widehat{J}(\alpha) := -\omega^{-1}(A), \quad \widehat{\omega}(A, \alpha) := \alpha(JA). \tag{9}$$

Keeping in mind that $\mathrm{DGA}(\mathfrak{h}, J)$ is an exterior algebra generated by $\mathfrak{k}^1 = \mathfrak{h}^{(1,0)} \oplus \mathfrak{h}^{*(0,1)}$, in [12] we construct a natural homomorphism of DGAs so that it induces a quasi-isomorphism.

Theorem 5.2 *There exists natural quasi-isomorphisms.*

$$\mathrm{DGA}(\widehat{\mathfrak{h}}, \widehat{\omega}) \approx \mathrm{DGA}(\mathfrak{h}, J), \quad \mathrm{DGA}(\mathfrak{h}, \omega) \approx \mathrm{DGA}(\widehat{\mathfrak{h}}, \widehat{J}).$$

In particular, Table (8) *is a table of weak mirror pairs.*

From Table (8), one notices that $\mathfrak{h}_4$ and $\mathfrak{h}_7$ form a mirror pair with different algebraic structures. However, readers are warned that this table only provides the underlying algebras and their dual semi-direct products. It does not specify the pseudo-Kähler structures. The following is a non-trivial result based on analysis in [13].

Theorem 5.3 *Let (J, ω) be a special Lagrangian structure on $\mathfrak{h}_\ell$, with $\ell = 1$, 8, 9, 10. Then $\widehat{\mathfrak{h}}_\ell \cong \mathfrak{h}_\ell$ and* $\mathrm{DGA}(\mathfrak{h}_\ell, J) \approx \mathrm{DGA}(\mathfrak{h}_\ell, \omega) \approx \mathrm{DGA}(\widehat{\mathfrak{h}}, \widehat{\omega}) \approx \mathrm{DGA}(\widehat{\mathfrak{h}}, \widehat{J})$.

It is interesting to point out that the above theorem does not apply to $\mathfrak{h}_{11}$ although $\widehat{\mathfrak{h}}_{11} \cong \mathfrak{h}_{11}$. A rather subtle analysis in [13] proves that $\mathrm{DGA}(\mathfrak{h}_{11}, J) \not\approx \mathrm{DGA}(\mathfrak{h}_{11}, \omega)$ when (J, ω) is a pseudo-Kähler structure on $\mathfrak{h}_{11}$. Since there is a large family of special Lagrangian pseudo-Kähler structures on $\mathfrak{h}_{11}$ [12], through Table (8) we encounter all possibilities on special Lagrangian nilpotent algebras $(\mathfrak{h}, J, \omega)$ and its mirror $(\widehat{\mathfrak{h}}, \widehat{J}, \widehat{\omega})$.

- Self mirror: $\mathrm{DGA}(\mathfrak{h}, J) \approx \mathrm{DGA}(\mathfrak{h}, \omega) \approx \mathrm{DGA}(\widehat{\mathfrak{h}}, \widehat{\omega}) \approx \mathrm{DGA}(\widehat{\mathfrak{h}}, \widehat{J})$ and $\mathfrak{h} \cong \widehat{\mathfrak{h}}$.
- There exists a nilpotent algebra $\mathfrak{h}$ with special Lagrangian pseudo-Kahler structure (ω, J) such that $\widehat{\mathfrak{h}} \cong \mathfrak{h}$ but $(\mathfrak{h}, \omega, J)$ is not isomorphic to $(\widehat{\mathfrak{h}}, \widehat{\omega}, \widehat{J})$.
- There are weak mirror pairs with non-isomorphic Lie algebras.

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