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# Kähler surfaces with zero scalar curvature

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**Abstract.** The equations of Euler type used by Belinskii *et al* to generate Ricci-flat Kähler metrics on 4-space is generalized to be a system of equations that generates Kähler metrics with vanishing scalar curvature. This family of new metrics also contains a family of scalar-flat Kähler surfaces constructed by LeBrun.

## 1. Introduction

The most famous Kähler metric with zero scalar curvature on a compact complex surface is the Calabi–Yau metric on a K3-surface. There is also the conformally flat metric on the local product of a rational curve and a curve with higher genus. In fact, for algebraic reasons, it is known that a Kähler metric has vanishing scalar curvature if and only if the surface is anti-self-dual with respect to the canonical orientation, see e.g. Itoh (1984)—note that in this paper a metric with (anti-) self-dual Weyl curvature is called an (anti-) self-dual metric or an (anti-) self-dual surface. Coupling this fact with various Bochner type arguments, several authors (Boyer 1986, Itoh 1984, LeBrun 1986) were able to find that if $(M, g)$ is a compact Kähler surface with vanishing scalar curvature, then $(M, g)$ is either conformally flat or the K3-surface with the Calabi–Yau metric or the blowing-up of a rational surface.

On this last type of surfaces, it is not known whether there is such a metric. Therefore, the Calabi–Yau metric is the only known example of a conformally non-flat Kähler metric with vanishing scalar curvature on a compact surface. To construct new examples on rational surfaces, one may attempt to construct a twistor space (Besse 1987). This space is going to be a compact complex 3-fold with algebraic dimension zero (Poon 1990) yet foliated by rational curves.

If one is willing to give up the compactness of the surface, there is a list of interesting examples. First of all, there is the Eguchi–Hanson metric on the total space of the canonical bundle over the rational curve. Then there is a family of Ricci-flat examples constructed by Belinskii *et al* (1978). This family includes the Eguchi–Hanson metric. Also, Burns (1986) found that the standard metric on the blow-up of $\mathbf{C}^2$ at the origin is a Kähler metric with zero scalar curvature. This metric and the Eguchi–Hanson metric are both members of a new family of such metrics constructed by LeBrun (1988). Although the Belinskii–Gibbons–Page–Pope (BGPP) family and the LeBrun family are different and are constructed by slightly different methods, each member in these two families shares the following properties:

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P1 Bianchi type IX metric, i.e. it has an $SU(2)$ isometry group acting transitively on the 3-sphere (Landau and Lifshitz 1975).
P2 Kählerian.
P3 vanishing scalar curvature.
P4 asymptotically Euclidean.

We shall present a family of metrics possessing these four properties. Our family of metrics include the BGPP family and the LeBrun family. The BGPP family will be distinguished by the fact that they are the only Einstein metrics in this family. The LeBrun family will be distinguished by the fact that they are the only conformally Einstein non-Einstein metrics. Also, the family of metrics by LeBrun contains certain special cases which are complete while the general 3-axial metric in the BGPP family is incomplete as we expect our family is. All these metrics are determined by some very interesting ordinary differential equations.

## 2. The Bianchi type IX metric and the ODE

A metric of Bianchi type IX is a Riemannian metric on $\mathbf{R}^4\backslash\{0\}$ expressed in polar coordinates as follows:

$$g = (abc)^2 \mathrm{d}t^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2 \tag{2.1}$$

where $\{\sigma_1, \sigma_2, \sigma_3\}$ is the standard basis of left invariant 1-forms on $SU(2)$, $t$ is the coordinate on $\mathbf{R}_+$ and $a, b, c$ are smooth functions of $t$. When

$$e_0 = abc\mathrm{d}t \qquad e_1 = a\sigma_1 \qquad e_2 = b\sigma_2 \qquad e_3 = c\sigma_3$$

then $\{e_0, e_1, e_2, e_3\}$ forms an orthonormal coframe. We shall choose this order to be the orientation. When $(\omega_{ij})_{0\leq i,j\leq 3}$ is the connection matrix with respect to this coframe, then

$$\begin{aligned}
\omega_{01} &= -\frac{1}{abc}\frac{\dot{a}}{a}e_1 & \qquad \omega_{12} &= \frac{1}{2abc}(c^2 - a^2 - b^2)e_3 \\
\omega_{02} &= -\frac{1}{abc}\frac{\dot{b}}{b}e_2 & \qquad \omega_{13} &= \frac{1}{2abc}(a^2 + c^2 - b^2)e_2 \\
\omega_{03} &= -\frac{1}{abc}\frac{\dot{c}}{c}e_3 & \qquad \omega_{23} &= \frac{1}{2abc}(a^2 - b^2 - c^2)e_1
\end{aligned}$$

where an overdot means $\mathrm{d}/\mathrm{d}t$.

Belinskii *et al* found that if they set

$$\omega_{01} = \omega_{23} \qquad \omega_{02} = -\omega_{13} \qquad \omega_{03} = \omega_{12} \tag{2.2}$$

then the system of ODE in $a, b, c$ can be solved. By setting

$$a^2 = \frac{\omega_2\omega_3}{\omega_1} \qquad b^2 = \frac{\omega_3\omega_1}{\omega_2} \qquad c^2 = \frac{\omega_1\omega_2}{\omega_3} \tag{2.3}$$

the equations (2.2) are transformed to be

$$\begin{cases} \dot{\omega}_1 = \omega_2\omega_3 \\ \dot{\omega}_2 = \omega_3\omega_1 \\ \dot{\omega}_3 = \omega_1\omega_2 . \end{cases} \tag{2.4}$$

The BGPP family is obtained by explicitly solving this system of ODE by elliptic functions.

We consider a Bianchi type IX metric determined by (2.1) and (2.3) where the $\omega$ are a solution of

$$\begin{cases} \dot{\omega}_1 = \omega_2\omega_3 + \alpha\omega_1 \\ \dot{\omega}_2 = \omega_3\omega_1 + \beta\omega_2 \\ \dot{\omega}_3 = \omega_1\omega_2 + \gamma\omega_3 \end{cases} \tag{2.5}$$

and we ask which kind of geometry may be obtained.

To compute the curvature, we choose a basis for $\Lambda^2$

$$\{\xi_1^+, \xi_2^+, \xi_3^+, \xi_1^-, \xi_2^-, \xi_3^-\}$$

where $(\xi_1^\pm, \xi_2^\pm, \xi_3^\pm)$ spans the (anti-) self-dual 2-forms $\Lambda^2_\pm$:

$$\begin{aligned} \xi_1^+ &= e_0 \wedge e_1 + e_2 \wedge e_3 \qquad & \xi_1^- &= e_0 \wedge e_1 - e_2 \wedge e_3 \\ \xi_2^+ &= e_0 \wedge e_2 - e_1 \wedge e_3 \qquad & \xi_2^- &= e_0 \wedge e_2 + e_1 \wedge e_3 \\ \xi_3^+ &= e_0 \wedge e_3 + e_1 \wedge e_2 \qquad & \xi_3^- &= e_0 \wedge e_3 - e_1 \wedge e_2 . \end{aligned}$$

With respect to this frame, the curvature has the following block form (Besse 1987):

$$R = \begin{pmatrix} A & B \\ B^{\mathrm{T}} & D \end{pmatrix}$$

where $s = 4 \operatorname{trace} D$ is the scalar curvature, $W^+ = A - \frac{1}{12}s, W^- = D - \frac{1}{12}s$ are the self-dual and anti-self-dual parts of the Weyl tensor and $B = \text{Ricci} - \frac{1}{4}s$ is the trace-free Ricci tensor.

*Lemma 1.*

(i) $\operatorname{trace} D = (\lambda\mu + \mu\nu + \nu\lambda)/2(abc)^2$.

(ii) $D = [1/2(abc)^2] \operatorname{diag}(\lambda\mu - \mu\nu + \nu\lambda; \lambda\mu + \mu\nu - \nu\lambda; -\lambda\mu + \mu\nu + \nu\lambda)$.

(iii) $B = \operatorname{diag}(-\lambda/(bc)^2 + \operatorname{trace} D, -\mu/(ca)^2 + \operatorname{trace} D, -\nu/(ab)^2 + \operatorname{trace} D)$ where

$$-2\lambda = \gamma + \beta - \alpha \qquad -2\mu = \alpha + \gamma - \beta \qquad -2\nu = \alpha + \beta - \gamma . \tag{2.6}$$

*Proof.* Elementary calculation. □

*Corollary 1.* The metric is self-dual with vanishing scalar curvature if and only if

$$\lambda\mu = \mu\nu = \nu\lambda = 0.$$

It is Ricci flat if and only if

$$\lambda = \mu = \nu = 0.$$

As a Kähler metric with vanishing scalar curvature is anti-self-dual, we shall look for self-dual metrics for the moment and hence we shall assume that our metric is self-dual with vanishing scalar curvature. Without loss of generality, we assume that

$$\lambda = \mu = 0.$$

i.e. $\alpha = \beta, \gamma = 0$. Then the system (2.4) is reduced to

$$\begin{cases} \dot{\omega}_1 = \omega_2\omega_3 + \alpha\omega_1 \\ \dot{\omega}_2 = \omega_3\omega_1 + \alpha\omega_2 \\ \dot{\omega}_3 = \omega_1\omega_2. \end{cases} \tag{2.7}$$

## 3. The LeBrun metric

From corollary 1, we see that the BGPP metrics are the only Ricci flat metrics in our family. Let us consider another special case when $\omega_1^2 = \omega_2^2$. As $a^2$ and $b^2$ are required to be positive, $\omega_1 = \omega_2$. The system (2.7) is reduced to

$$\begin{cases} \dot{\omega}_1 = \omega_1\omega_3 + \alpha\omega_1 \\ \dot{\omega}_3 = \omega_1^2. \end{cases}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\omega_1^2 - \omega_3^2 - 2\alpha\omega_3) = 0.$$

i.e.

$$\omega_1^2 - \omega_3^2 - 2\alpha\omega_3 - k = 0$$

for some constant $k$. I.e.

$$\dot{\omega}_3 - \omega_3^2 - 2\alpha\omega_3 - k = 0.$$

As this equation is simply a Riccati equation, it can be solved explicitly. However, to recognize the metric, we simply take

$$r^2 = 4\omega_3.$$

Note that for appropriate choice of initial condition $\omega_3$, as a solution to the Riccati equation, can be a function with positive value varying from zero to infinity.

Then from (2.3), we have

$$a^2 = b^2 = \tfrac{1}{4}r^2 \qquad c^2 = \tfrac{1}{4}r^2 W$$

where $W = 1 + 8\alpha/r^2 + 16k/r^4$. As $2r\mathrm{d}r = 4\dot{\omega}_3 \mathrm{d}t$

$$(abc)^2 \mathrm{d}t^2 = (1/W)\mathrm{d}r^2.$$

Therefore, the metric can be expressed as

$$g = \mathrm{d}r^2/W + \tfrac{1}{4}r^2(V\sigma_1^2 + V^{-1}\sigma_2^2 + W\sigma_3^2) \tag{3.1}$$

with $W$ given as above and $V = 1$. This is precisely the metric obtained by LeBrun (1988).

We have not been able to solve the system (2.7) explicitly except in this case and in the case when $\alpha = 0$. However, in the next two sections, we find an expression for the metric that generalizes (3.1). Also, we prove that the system of ODE (2.7) induces the properties P1–P4 of the metric.

## 4. The Kählerian property

We shall now see how the fact that a solution to the system of ODE (2.7) is encoded in the metric, implies the Kählerian property of the metric.

*Proposition 1.* If the metric (2.1) is determined by a solution to (2.7) and the equations (2.3), then it is a Kähler metric.

*Proof.* An almost complex structure $I$ is defined by

$$I(\omega_3 \mathrm{d}t) = \sigma_3 \qquad I(\omega_2\sigma_1) = \omega_1\sigma_2.$$

Then

$$\Omega_1 = \omega_3 \mathrm{d}t + \mathrm{i}\sigma_3 \qquad \Omega_2 = \omega_2\sigma_1 + \mathrm{i}\omega_1\sigma_2$$

are (1,0)-forms and

$$g = \frac{\omega_1\omega_2}{\omega_3}\Omega_1 \otimes \overline{\Omega}_1 + \frac{\omega_3}{\omega_1\omega_2}\Omega_2 \otimes \overline{\Omega}_2.$$

The Kähler form of this almost Hermitian metric is

$$\begin{aligned}\Omega &= \frac{\omega_1\omega_2}{\omega_3}\omega_3 \mathrm{d}t \wedge \sigma_3 + \frac{\omega_3}{\omega_1\omega_2}\omega_1\omega_2\sigma_1 \wedge \sigma_2 \\ &= \omega_1\omega_2 \mathrm{d}t \wedge \sigma_3 + \omega_3\sigma_1 \wedge \sigma_2 \\ &= \omega_1\omega_2 \mathrm{d}t \wedge \sigma_3 + \omega_3 \mathrm{d}\sigma_3.\end{aligned}$$

since $\mathrm{d}\sigma_i = \frac{1}{2}\varepsilon_{ijk}\sigma_j \wedge \sigma_k$.

It is a closed form because

$$d\Omega = -\omega_1\omega_2 dt \wedge d\sigma_3 + \dot{\omega}_3 dt \wedge d\sigma_3 = 0$$

by the last equation in the system (2.7).

To finish the proof of the proposition, it suffices to show that the ideal generated by $\Omega_1, \Omega_2$ is closed under exterior differentiation. But as a consequence of the first two equations in (2.7)

$$d\Omega_2 = -(\omega_1\sigma_1 + i\omega_2\sigma_2) \wedge \Omega_1 + \alpha dt \wedge \Omega_2.$$

Also, by the nature of the left invariant 1-form on $SU(2)$

$$d\Omega_1 = \frac{1}{\omega_1}\sigma_1 \wedge \Omega_2. \qquad \square$$

*Remark.* Note that the canonical orientation with respect to the complex structure $I$ is determined by the ordered basis

$$\{\omega_3 dt, \omega_2\sigma_1, \sigma_3, \omega_1\sigma_2\}.$$

so with respect to this orientation the metric is anti-self-dual.

## 5. Asymptotic behaviour

Let

$$f_1 = e^{-\alpha t}\omega_1 \qquad f_2 = e^{-\alpha t}\omega_2 \qquad f_3 = \omega_3$$

then (2.7) is transformed into

$$\begin{cases} \dot{f}_1 = f_2 f_3 \\ \dot{f}_2 = f_3 f_1 \\ \dot{f}_3 = e^{2\alpha t} f_1 f_2. \end{cases} \tag{5.1}$$

Therefore, $f_1^2 - f_2^2$ is a constant. We may assume that this is a positive constant:

$$f_1^2 - f_2^2 = A^2.$$

The case when $A = 0$ was already discussed in section 2. We shall assume that $A \neq 0$ and that $\alpha \neq 0$.

If $G$ is a function such that

$$f_1 - f_2 = Ae^{-G}$$

then

$$f_1 + f_2 = Ae^{G}$$

and hence

$$f_1 = A\cosh G \qquad f_2 = A\sinh G. \tag{5.2}$$

Since

$$-\dot{G} = \frac{\mathrm{d}}{\mathrm{d}t}\ln(f_1 - f_2) = \frac{\dot{f}_1 - \dot{f}_2}{f_1 - f_2} = -f_3$$

we get

$$f_3 = \dot{G}. \tag{5.3}$$

Combining (4.1), (5.2) and (5.3), we have

$$\ddot{G} = \tfrac{1}{2}A^2 \mathrm{e}^{2\alpha t}\sinh 2G.$$

*Remark.* When $\alpha = 0$ this second order differential equation has been studied extensively by many authors (Wente 1986, Barouch *et al* 1973a,b, Perram and Barber 1974) and is referred to as the sinh Gordon equation or the Poisson–Boltzmann equation.

As in section 2, let $r^2 = 4\omega_3$, i.e.

$$r^2 = 4\dot{G}. \tag{5.4}$$

Then

$$a^2 = \tfrac{1}{4}r^2 V \qquad b^2 = \tfrac{1}{4}r^2 V^{-1} \qquad c^2 = \tfrac{1}{4}r^2 W$$

where $V = \tanh G, W = 8A^2 r^{-4}\mathrm{e}^{2\alpha t}\sinh 2G$, and the metric can be expressed as

$$\mathrm{d}r^2/W + \tfrac{1}{4}r^2(V\sigma_1^2 + V^{-1}\sigma_2^2 + W\sigma_3^2). \tag{5.5}$$

It can be seen as a 3-axial generalization of (3.1).

To justify the implicit coordinate change in (5.4), we have to check that $G$ is a strictly increasing function.

*Lemma 2.* Let $(\eta, \zeta)$ be the maximal interval on which the initial valued problem

$$\begin{cases} \ddot{G} = \tfrac{1}{4}A^2\mathrm{e}^{2\alpha t}(\mathrm{e}^{2G} - \mathrm{e}^{-2G}) \\ G(0) = u > 0 \\ \dot{G}(0) = v > 0 \end{cases}$$

has an analytic solution. Then $G$ is monotonically blowing up to infinity as $t \to \zeta$.

*Proof.* We can actually see that

$$G > 0, \dot{G} > 0, \ddot{G} > h\mathrm{e}^{2\alpha t}$$

for all $t$ in $(0, \zeta)$, where $h$ is a positive constant.

This initial value problem always has a unique analytic solution in a neighbourhood of the origin. As $u$ and $v$ are positive, $G, \dot{G}$ are positive in a neighbourhood of the origin as well. Let $t_0$ be the first $t$ in $(0, \zeta)$ at which $G$ vanishes. This is precisely the first $t$ at which $\ddot{G}$ vanishes, and hence is the first critical point of $\dot{G}$.

On the other hand, as $G(0) > 0$ and $\dot{G}(0) > 0$, when $G(t_0) = 0$, there is a $t_1$ in the interval $(0, t_0)$ such that $\dot{G}(t_1) = 0$ and hence there is a $t_2$ in the interval $(0, t_1)$ such that $\ddot{G}(t_2) = 0$. But this is a contradiction to the definition of $t_0$. It follows that $G$ is strictly positive for all $t \in (0, \zeta)$.

As $\mathrm{e}^{2G} - \mathrm{e}^{-2G}$ is a strictly increasing function of $G$, then

$$\ddot{G} > \tfrac{1}{4}A^2\mathrm{e}^{2\alpha t}(\mathrm{e}^{2u} - \mathrm{e}^{-2u}) > 0$$

for all $t \in (0, \zeta)$. Now it is obvious that $G$ will blow up to infinity as $t$ approaches to $\zeta$. □

From this it follows that $g$ is asymptotically flat: Since $V = \tanh G$ and $G$ is blowing up to infinity, $V \to 1$. Meanwhile, as $f_1 - f_2 = Ae^{-G}$, when $G$ is blowing up to infinity, the solution is approximated by the solution of LeBrun which is asymptotically Euclidean. To be more precise, $W = \ddot{G}/\dot{G}^2$

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}r} &= \frac{\mathrm{d}W}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r}\\ &= \left(\frac{\dddot{G}}{\dot{G}^2} - 2\frac{\ddot{G}^2}{\dot{G}^3}\right)\left(\frac{r}{2\ddot{G}}\right)\\ &= \frac{\alpha r}{\dot{G}^2} + \frac{r}{\dot{G}}\coth 2G - r\frac{\ddot{G}}{\dot{G}^3}\\ &= \frac{16\alpha}{r^3} - \frac{4}{r}W + \frac{4}{r}\coth 2G.\end{aligned}$$

Since $\coth 2G \to 1$ when $r \to \infty$, asymptotically, $W$ is a solution of

$$\frac{\mathrm{d}W}{\mathrm{d}r} + \frac{4}{r}W = \frac{16\alpha}{r^3} + \frac{4}{r}.$$

Thus, asymptotically

$$W = 1 + \frac{8\alpha}{r^2} + \frac{16k}{r^4}$$

for some constant $k$. In particular, $W \to 1$ as $r \to \infty$. Therefore, our metric is asymptotically Euclidean.

## 6. A conformal change

From corollary (1.8), we know that the BGPP metrics are the only Ricci-flat metrics in our family. As our metrics have zero scalar curvature, the BGPP metrics are the only Einstein metrics in our family. We would like to know if there is an Einstein metric within the conformal class of our family. It turns out that this is the geometric condition that singles out the BGPP family and the LeBrun family.

*Proposition 2.* A metric determined by system (2.7) is conformally equivalent to an Einstein metric of cohomogeneity one, if and only if it is the BGPP metric or the LeBrun metric.

*Proof.* Let $Z$ be the trace-free part of the Ricci tensor of our metric. When $\hat{g} = \phi^{-2}g$ is an Einstein metric, then (Besse 1987):

$$0 = Z\phi + 2(\nabla d\phi + \tfrac{1}{4}\Delta\phi g). \tag{6.1}$$

where the covariant derivative and the Laplacian are associated with our metric.

From lemma 1, we find that

$$Z = \frac{16\alpha}{r^4}(e_0^2 - e_1^2 - e_2^2 + e_3^2).$$

When the metric is given in the form of (4.5), the covariant derivative and Laplacian can be worked out as follows:

$$\nabla d\phi = \left(\ddot{\phi}W + \frac{1}{2}\dot{\phi}\dot{W}\right)e_0^2 + \frac{\dot{\phi}W}{r}\left(1 + \frac{r\dot{V}}{2V}\right)e_1^2 + \frac{\dot{\phi}W}{r}\left(1 - \frac{r\dot{V}}{2V}\right)e_2^2 + \frac{\dot{\phi}W}{r}\left(1 + \frac{r\dot{W}}{2W}\right)e_3^2$$

$$\Delta\phi = -\left(\ddot{\phi}W + \dot{\phi}\dot{W} + 3\dot{\phi}\frac{W}{r}\right)$$

where the overdot means d/d$r$. Here the property of cohomogeneity one has forced $\phi$ to depend only on $r$. Then (5.2) is a system of four equations:

$$0 = \frac{16\alpha}{r^4}\phi + 2\left(\ddot{\phi}W + \frac{1}{2}\dot{\phi}\dot{W}\right) - \frac{1}{2}\left(\ddot{\phi}W + \dot{\phi}\dot{W} + 3\dot{\phi}\frac{W}{r}\right) \tag{6.2}$$

$$0 = -\frac{16\alpha}{r^4}\phi + 2\frac{\dot{\phi}W}{r}\left(1 + \frac{r\dot{V}}{2V}\right) - \frac{1}{2}\left(\ddot{\phi}W + \dot{\phi}\dot{W} + 3\dot{\phi}\frac{W}{r}\right) \tag{6.3}$$

$$0 = -\frac{16\alpha}{r^4}\phi + 2\frac{\dot{\phi}W}{r}\left(1 - \frac{r\dot{V}}{2V}\right) - \frac{1}{2}\left(\ddot{\phi}W + \dot{\phi}\dot{W} + 3\dot{\phi}\frac{W}{r}\right) \tag{6.4}$$

$$0 = \frac{16\alpha}{r^4}\phi + 2\left(\frac{\dot{\phi}W}{r} + \frac{1}{2}\dot{\phi}\dot{W}\right) - \frac{1}{2}\left(\ddot{\phi}W + \dot{\phi}\dot{W} + 3\dot{\phi}\frac{W}{r}\right). \tag{6.5}$$

Subtracting (6.3) from (6.4), we have

$$\dot{\phi}W\dot{V}/V = 0.$$

This is possible if and only if $\dot{\phi} = 0$ or $\dot{V} = 0$. In the first case, the metric is exactly the BGPP metric and it is already Ricci-flat. In the second case, the metric is the LeBrun metric. We have to check that the conformal class of the LeBrun metric does contain an Einstein metric.

Subtracting (6.2) from (6.5), we have

$$0 = \ddot{\phi} - \dot{\phi}/r.$$

Then $\phi = cr^2 + h$ for some constants $c, h$. From section 3, we have already seen that, in this case, $W = 1 + 8\alpha/r^2 + 16k/r^4$. Subsituting this $W$ into the equations above, we find that $\phi = r^2 + 4k/\alpha$ is the only solution. □

*Remark.* Consider the conformal class of the metric by LeBrun again: the metric is given by

$$\begin{cases} g = \mathrm{d}r^2/W + \frac{1}{4}r^2(\sigma_1^2 + \sigma_2^2 + W\sigma_3^2) \\ W = 1 + A/r^2 + B/r^4. \end{cases} \tag{6.6}$$

Multiply the metric with

$$\psi^2 = 4(4B - A^2)(Ar^2 + 2B)^{-2}. \tag{6.7}$$

It is easily seen that $\psi$ satisfies the differential equation

$$\Delta\psi = A\psi^3 \tag{6.8}$$

where

$$\Delta\psi = -\psi'' W - \psi'(W' + 3Wr^{-1})$$

is the Laplacian of the LeBrun metric. Thus, the LeBrun metric is conformal to a metric with constant scalar curvature $\neq 0$. Indeed, the metric $\psi^2 g$ is equal to the Einstein metric:

$$\psi^2 g = \frac{1}{(\varepsilon - m^2R)^2}\left\{\frac{m^2(1+\varepsilon R)}{(1+m^2R^2)R}dR^2 + m^2R(1+\varepsilon R)(\sigma_1^2+\sigma_2^2) + \frac{m^2R(1+m^2R^2)}{1+\varepsilon R}\sigma_3^2\right\} \tag{6.9}$$

where

$$A = -2\varepsilon \qquad B = \varepsilon^2 + m^2 \qquad r^2 = \frac{1+\varepsilon R}{R}.$$

This metric was found in Pedersen (1986) but actually lies in the family of type $D$ metrics given by Gibbons and Pope (1978) and is a Riemannization of the Taub–NUT de Sitter metric.

Now, the family of metrics in (6.9) contains Einstein metrics with positive scalar curvature and metrics with negative scalar curvature. In particular, if we put

$$M = im \qquad \varepsilon = M^2 \qquad \rho^2 = R$$

and multiply the metric with $k = \frac{1}{4}M^2$, then we obtain the following metric on the ball in $\mathbf{R}^4$:

$$\mathrm{d}s^2 = \frac{1}{(1-\rho^2)^2}\left\{\frac{1-M^2\rho^2}{1-M^2\rho^4}\mathrm{d}\rho^2 + \frac{\rho^2}{4}(1-M^2\rho^2)(\sigma_1^2+\sigma_2^2) + \frac{\rho^2}{4}\frac{1-M^2\rho^4}{1-M^2\rho^2}\sigma_3^2\right\}. \tag{6.10}$$

This is the Einstein metric obtained by Pedersen via LeBrun's H-space construction (LeBrun 1982) from a conformal structure on the 3-sphere at infinity given by

$$\sigma_1^2 + \sigma_2^2 + I_3\sigma_3^2$$

where

$$I_3 = \frac{1}{1-M^2} \geq 1.$$

The metric by LeBrun (5.8) where

$$A = (n-2)a^2 \qquad B = -(n-1)a^4$$

is regular on the total space of the bundle $\mathcal{O}(-n) \to \mathbf{CP}^1$ and if

$$a^2 = \frac{2(n-2)}{n^2} \qquad (n \geq 3)$$

this metric is conformal to the Einstein metric (6.10) where

$$I_3 = \frac{n^2}{4(n-1)}.$$

## 7. Summary and outstanding problems

We have proved the following

*Theorem 1.* Let $g$ be a Bianchi type IX metric

$$g = (abc)^2 \mathrm{d}t^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2$$

and let

$$a^2 = \frac{\omega_2\omega_3}{\omega_1} \qquad b^2 = \frac{\omega_3\omega_1}{\omega_2} \qquad c^2 = \frac{\omega_1\omega_2}{\omega_3}.$$

Suppose that $(\omega_1, \omega_2, \omega_3)$ satisfies the system of ODE

$$\dot{\omega}_1 = \omega_2\omega_3 + \alpha\omega_1 \qquad \dot{\omega}_2 = \omega_3\omega_1 + \alpha\omega_2 \qquad \dot{\omega}_3 = \omega_1\omega_2.$$

Then, the metric $g$ is a Kähler metric with zero scalar curvature. The metric is asymptotically Euclidean and may be represented in the following way:

$$g = \mathrm{d}r^2/W(r) + \tfrac{1}{4}r^2(V(r)\sigma_1^2 + V(r)^{-1}\sigma_2^2 + W(r)\sigma_3^2)$$

and is a 3-axial generalization of the LeBrun metric which corresponds to $V = 1$, $W = 1 + 8\alpha/r^2 + 16k/r^4$. The LeBrun metric is the only conformally Einstein non-Einstein metric in our family of metrics, which also contains the Ricci-flat metric by Belinskii *et al* .

*Remark.* In the description of the hyper-Kähler metric on the moduli space of monopoles Atiyah and Hitchin (1988) worked on a relation between a Bianchi IX metric and a system of ODE similar to ours.

In the remaining paragraphs we shall mention some outstanding problems related to what has been said above.

*Remark.* To us, the generalization (2.7) is by no means natural. It is simply a generalization that works. In fact, one may consider the system

$$\begin{cases} \dot{\omega}_1 = \omega_2\omega_3 + E\beta\omega_2 \\ \dot{\omega}_2 = \omega_3\omega_1 + E\beta k^2\omega_1 \\ \dot{\omega}_3 = \omega_1\omega_2. \end{cases}$$

When $\beta = 0$, a solution corresponds to the BGPP metrics. When $k = 1$, and $\omega_1 = \omega_2$, a solution corresponds to the LeBrun metrics. Therefore, one may ask if this new family of metrics can be a generalization that we want. This system can be solved explicitly by elliptic functions and the metric can also be found explicitly as follows:

$$\mathrm{d}r^2/W + \tfrac{1}{4}r^2(V\sigma_1^2 + V^{-1}\sigma_2^2 + W\sigma_3^2)$$

where

$$W = \left[\left(1 + \frac{8E\beta}{r^2} - \frac{16E^2}{r^4}\right)\left(1 + \frac{8E\beta k^2}{r^2} - \frac{16E^2k^2}{r^4}\right)\right]^{1/2}$$

$$V = \left[\left(1 + \frac{8E\beta k^2}{r^2} - \frac{16E^2k^2}{r^4}\right)\left(1 + \frac{8E\beta}{r^2} - \frac{16E^2}{r^4}\right)^{-1}\right]^{1/2}.$$

However, this metric is neither self-dual nor has it vanishing scalar curvature except when it is the BGPP metric or the LeBrun metric.

This example shows that one has to understand how to describe the system (2.7) or the corresponding family of metrics naturally as a generalization of the BGPP metrics and the LeBrun metric.

*Remark.* Since our metric is anti-self-dual, it corresponds to a twistor space. A description of this twistor space may explain the generalization (2.7).

Since LeBrun's metric is conformal to the Einstein metric in (6.9) the twistor space of these 2-axial metrics has already been described in Pedersen (1986) as a line bundle over an open set in the quadric surface. Also, Galicki (1987) has rediscovered the metric in (6.10) via quaternionic Kähler reduction of the dual of $\mathbf{HP}^2$. Thus, the twistor space may also be described as a reduction of the six-dimensional twistor space of the dual of $\mathbf{HP}^2$. It is tempting also to try to obtain 3-axial metrics via quaternionic reduction.

We hope to discuss some of these subjects in the future.

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*Note added in proof.* After the completion of this paper Paul Tod pointed out to us that the differential equation

$$\ddot{G} = \tfrac{1}{2}A^2 e^{2\alpha t} \sinh 2G$$

is equivalent to a third Painlevé equation

$$w^{-1}w_{zz} - w^{-2}w_z^2 + z^{-1}w^{-1}w_z - \tfrac{1}{4}A^2(w^2 - w^{-2})$$

where $\alpha = 0 = \beta$, $\delta = -\gamma = -\frac{1}{4}A^2$. To see this, make the substitution $z = \alpha^{-1}e^{\alpha t}$ to get the rotational invariant sinh Gordon equation

$$z^{-1}(zG_z)_z = \tfrac{1}{2}A^2 \sinh 2G.$$

Then, set $w = e^G$ to get the Painlevé form. In fact, for this subclass of the third Painlevé equation the Painlevé transcendents take a special form.

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